A stratification of the space of all k-planes in $\mathbb{C}^n$

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Abstract

To each $k \times n$ matrix M of rank k, we associate a *juggling pattern* of periodicity n with k balls. The juggling pattern actually only depends on the k-plane spanned by the rows, so gives a decomposition of the "Grassmannian" of all k-planes in n-space.

There are many connections between the geometry and the juggling. For example, the natural topology on the space of matrices induces a partial order on the space of juggling patterns, which indicates whether one pattern is "more excited" than another.

This same decomposition turns out to naturally arise from totally positive geometry [Lusztig 1994, Postnikov ~2004], characteristic p geometry [Knutson-Lam-Speyer 2011], and noncommutative geometry [Brown-Goodearl-Yakimov 2005]. It also arises by projection from the manifold of full flags in n-space, where there is no cyclic symmetry.

These transparencies are available at `http://math.cornell.edu/~allenk/`

A discrete invariant of matrices.

For the purposes of this talk, an **invariant of matrices** is a function `f : {matrices} → somewhere` that is invariant under row operations, or equivalently, $f(M) = f(AM)$ for A invertible. One of the best known is $\mathrm{rank} : M_{k\times n} \to \mathbb{N}$ (which is also invariant under column operations).

Today's is the following. Think of M as a list $\vec{v}_1, \ldots, \vec{v}_n$ of k-dimensional column vectors, and extend it to be an infinite but periodic list, $\vec{v}_i = \vec{v}_{n+i}$. Then define

$$J_M : \mathbb{Z} \to \mathbb{Z}, \qquad J_M(i) := \min\left\{ j \geq i : \vec{v}_i \in \mathrm{span}(\vec{v}_{i+1}, \ldots, \vec{v}_j) \right\} \quad \leq i + n.$$

For example,

$$\begin{array}{cccccccccc} \boldsymbol{v}_{-1} & \boldsymbol{v}_0 & \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 & \boldsymbol{v}_5 & \boldsymbol{v}_6 & \boldsymbol{v}_7 & \boldsymbol{v}_8 \end{array}$$

$$\cdots \begin{matrix}0&0\\0&0\\1&0\end{matrix} \begin{bmatrix}1&0&0&0&0&0\\0&1&1&0&0&0\\0&0&0&1&1&0\end{bmatrix} \begin{matrix}1&0\\0&1\\0&0\end{matrix} \cdots$$

$$J(i)-i: \quad 6 \quad 1 \quad 5 \quad 1 \quad 5 \quad 0$$

A nonobvious property: J_M is 1:1 and onto! What else is true about these J_M?

Bounded juggling patterns, with a fixed periodicity n.

An **affine permutation** $J : \mathbb{Z} \to \mathbb{Z}$ is a function that's 1:1 and onto, with the periodicity $J(i+n) = J(i) + n \ \forall i$. These form a group isomorphic to $S_n \ltimes \mathbb{Z}^n$, where $S_n := Sym(\mathbb{Z}/n)$ is the finite permutation group.

If we try to interpret $i \mapsto J(i)$ as "A ball thrown at time i comes down at time $J(i) - \frac{1}{2}$, and is then thrown at time $J(i)$" we had better insist $J(i) \geq i$, so balls land *after they are thrown*. Call such affine permutations **juggling patterns**. The number of balls in the air at time $i + \frac{1}{2}$, $\#\left\{k < i + \frac{1}{2} : J(k) > i + \frac{1}{2}\right\}$, is finite and (thankfully) independent of i.

What jugglers actually make use of is not J, but its associated **siteswap** $J(1)-1 \ \ J(2)-2 \ \ldots \ J(n)-n$, the list of throw ~~heights~~ ~~durations~~ durations$+\frac{1}{2}$. Useful theorem to come: the number of balls is the average of the siteswap.

Some examples: $3 \sim 3333, 4, 1, 51, 441, 4413, 330, 4440, 42, 552, 51414, 53\ldots$
If you want to see another hour of this, look up "knutson juggling" on YouTube.

Define a **bounded juggling pattern** to be an affine permutation J that not only satisfies $J(i) \geq i$, but also $J(i) \leq i + n$, for all i.

Theorem [Postnikov ~2004, juggling interpretation in K-Lam-Speyer 2011]. Each J_M (from the last page) is a bounded juggling pattern, and every k-ball period-n bounded juggling pattern arises from some $k \times n$ matrices of rank k.

Total positivity of matrices.

Matrices with real entries in which every submatrix has nonnegative determinant have been studied since the 1930s and impact many areas (see the entire book [Karlin 1968]). In our context we consider real $k \times n$ matrices where every $k \times k$ submatrix has determinant ≥ 0. These have a surprising cyclic property, that will connect to the periodicity of our patterns:

Lemma. If $[\vec{v}_1 \cdots \vec{v}_n]$ is a totally nonnegative matrix, so is $[\vec{v}_2 \cdots \vec{v}_n\ (-1)^{k-1}\vec{v}_1]$.

These $\binom{n}{k}$ many $k \times k$ determinants are not independent; e.g. in 2×4 they satisfy

$$p_{13}\, p_{24} = p_{12}\, p_{34} + p_{14}\, p_{23}, \qquad p_{ij} := \det(\text{columns } i \text{ and } j)$$

which is very stringent if we also require each $p_{ij} \geq 0$!

Theorem [Postnikov ~2004]. Let $B(M) = \{S \subseteq \{1, \ldots, n\} : |S| = k, p_S \neq 0\}$, the **bases of the matroid** associated to the matrix M.
If M is totally nonnegative and rank k, then $B(M)$ and J_M determine each other, and $B(M)$ is called a **positroid**. (If $\mathrm{rank}(M) \neq k$, then $B(M) = \emptyset$.)
The **positroid $\mathbb{R}_{\geq 0}$-stratum** of totally nonnegative matrices with a given J_M is (nonempty and) homeomorphic to an open ball.

If one drops the total-nonnegativity assumption, the topology of a matroid stratum can be, in some senses, arbitrarily bad (Mnëv's universality theorem).

The Freshman's Dream, and splitting the Frobenius morphism.

Let R be a commutative ring in which $1 + 1 + \ldots + 1 = 0$, added up p times. If R has no zero divisors, then p must be prime. We assume p is prime and say that R has **characteristic** p.

The Freshman's Dream. In a ring of characteristic p, $(a + b)^p = a^p + b^p$, i.e. $r \mapsto r^p$ is an endomorphism called the **Frobenius**.

Call an abelian group homomorphism $\varphi : R \to R$ a **Frobenius splitting** if

- $\varphi(r^p) = r, \ \forall r \in R$ so, φ is a one-sided inverse
- $\varphi(r^p q) = r \ \varphi(q)$ another desirable property of such a "pth root" map.

Example. Let $R = \mathbb{F}_p[x]$, $\varphi(cx^k) = cx^{k/p}$ if $p \mid k$, 0 otherwise.
A similar rule works for $R = \mathbb{F}_p[x_1, \ldots, x_n]$, or that modulo any monomial ideal, and many other φ exist for these R.

Example. Let $R = \mathbb{F}_p[a^2, a^3] \leq \mathbb{F}_p[a]$, so $R \equiv \mathbb{F}_p[x, y]/\langle y^2 - x^3 \rangle$. Then $\nexists \varphi$.

It's easy to show that if R has a Frobenius splitting φ, then R must have no nilpotents. As the second example shows, though, the condition is much more stringent.

Compatibly split ideals.

In the category of "Frobenius split rings (R, φ) of characteristic p" the right notion of ideal $I \leq R$ is one such that $\varphi(I) \leq I$, called a **compatibly split ideal**.

Theorem [Enescu–Hochster 2008, Schwede 2009, Kumar–Mehta 2009]. If R is a Frobenius split Noetherian ring (or more generally a Noetherian scheme with a Frobenius splitting on its structure sheaf), then it has only finitely many compatibly split ideals (resp. ideal sheaves).

Sad proposition [K]. If $R = \mathbb{F}_p[x_{11}, \ldots, x_{kn}]$ is the functions on the space of $k \times n$ matrices, and $A = p_{12\cdots k}\, p_{23\cdots k+1}\, p_{34\cdots k+2} \cdots p_{n-1\ n\ 12\cdots k-2}\, p_{n12\cdots k-1}$, then for $n, k > 1, n \neq k$ there is no splitting φ that compatibly splits $\langle A \rangle$.

Luckily we don't want to apply this technology to *matrices*, but to rank k matrices up to row-equivalence. So some k columns $S \subseteq \{1, \ldots, n\}$ must form a basis, and we can use up the row operations making them the identity matrix.

Theorem [K-Lam-Speyer 2011]. Let R_S be the functions on the (affine) space of $k \times n$ matrices whose columns S are an identity matrix. Then there is a unique splitting on R_S that compatibly splits the $\langle A \rangle$ above, and its compatibly split prime ideals are exactly given by the positroid stratification.

This is more cleanly stated as being about a splitting on the **Grassmannian of k-planes in n-space**, which has an atlas given by these $\binom{n}{k}$ affine patches.

A noncommutative deformation of the Grassmannian.

Let R be a vector space, and $\cdot_\epsilon : R \times R \to R$ a family of associative products on it, one for each number ϵ. If $\cdot_0$ is commutative, then we can think of $(R, \cdot_0)$ as the ring of functions on a space $Spec\,(R, \cdot_0)$.

If $I \leq R$ is an ideal for every $\cdot_\epsilon$, then it is for $\cdot_0$, and defines a subset of $Spec\,(R, \cdot_0)$. But very few ideals arise this way, as noncommutative rings have far fewer of them than commutative rings do! One says that very few subvarieties "survive deformation to a noncommutative space".

$R = \mathbb{C}[x_{11}, \ldots, x_{kn}]$ has a family of products $\cdot_\epsilon$ described to first order by

$$x_{ij} \cdot_\epsilon x_{kl} = x_{kl} \cdot_\epsilon x_{ij} + \epsilon\, \mathrm{sign}(k-i)\mathrm{sign}(l-j) x_{il} x_{kj} + O(\epsilon^2)$$

Theorem [Brown-Goodearl-Yakimov 2006]. Let $I \leq R$ be a prime ideal of every $(R, \cdot_\epsilon)$, invariant under scaling the columns $(x_{ij} \mapsto t_j x_{ij})$. Then $I \leq (R, \cdot_0)$ defines one of our positroid strata, and each stratum arises this way from a unique I.

(This *is* connected to the Frobenius splitting, as follows. The first-order term above defines a *Poisson 2-tensor*, which wedged with some column-scaling vector fields gives an *anticanonical tensor*. From that tensor one can build a map $\phi : R \to R$, which may or may not be a splitting; in this case it is.)

An application of the positroid stratification to juggling.

Let $J, J' : \mathbb{Z} \to \mathbb{Z}$ be two juggling patterns. Call J' a **simple excitation** of J if

- $J(i) = J'(i)$ unless $i \equiv a, b \bmod n$ for some pair $a < b$
- $J(a) < J(b)$ and $J'(a) = J(b), J'(b) = J(a)$
- for all c in the open interval (a, b), $J(c) \notin (J(a), J(b))$.

Call J' an **excitation** of J if they are connected by a sequence of simple such. It is easy to see that J, J' must have the same number of balls, and their siteswaps must have the same average. Example (with a, b underlined):

$$\underline{51}414 \gtrdot 24\underline{41}4 \gtrdot 2\underline{42}34 \gtrdot 23334 \sim 333\underline{42} \gtrdot 33333$$

Proposition. The unique least excited pattern with k balls is $J(i) = i + k$, with all throws being ks. There are $\binom{n}{k}$ most excited bounded juggling patterns with k balls, with $(n - k)$ 0-throws and k n-throws.

Corollary (stated before): the average of the siteswap is the number of balls.

Theorem [K-Lam-Speyer 2011]. The positroid stratum for J' is in the closure of the stratum for J if and only if J' is an excitation of J.

Jugglers had already known about the $b = a+1$ simple excitations, but not these more general ones, nor that there is a well-defined **excitation number** given by the codimension of the corresponding stratum.

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