Schubert calculus puzzles from quiver varieties

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**Abstract**

In 2006 Paul Zinn-Justin observed that our puzzle rule [K-Tao ’03] for equivariant Schubert calculus on Grassmannians was based on an “R-matrix”, a solution to the Yang-Baxter equation. In 2017 Zinn-Justin and I extended this to discover and prove puzzle rules for K-theory of 2- and 3-step flag manifolds.

[Maulik and Okounkov ’12] trace R-matrices to Nakajima quiver varieties (whose definition I’ll recall), and I’ll explain how our puzzles can be seen directly from the quiver varieties. (In fact, the puzzles for a quiver variety extension are more symmetric!) We give a rule to recognize when a general-looking quiver variety is just $\mathbb{T}^*$ of a partial flag variety.

Then I’ll show a further extension, which was most easily discovered via the quiver variety interpretation, computing pullbacks in $K_{\mathbb{T}}$ along $\text{Fl}(\mathbb{C}^n) \rightarrow \text{Fl}(k, k+1, \ldots, n; \mathbb{C}^n) \times \text{Fl}(1, 2, \ldots, k; \mathbb{C}^n)$.

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An intersection theory problem.

Let $L_1, L_2$ be two different, but crossing, lines in 3-space. Let $Y_1, Y_2$ be the set of lines touching $L_1, L_2$ respectively. Then

$$Y_1 \cap Y_2 = \{\text{lines in the } L_1L_2 \text{ plane}\} \cup \{\text{lines through } L_1 \cap L_2\}$$

$$\{\text{lines doing both}\}$$

Let $\text{Gr}(1, \mathbb{P}^3) \cong \text{Gr}(2, \mathbb{C}^4)$ be the Grassmannian of lines in projective 3-space. Although $Y_1 \neq Y_2$ as sets, they are homologous in $\text{Gr}(2, \mathbb{C}^4)$, so define the same element “$S_{0101}$” in cohomology (or K-theory).

More generally, consider lines in $\mathbb{P}^{n-1}$ that touch a fixed $j$-plane and are contained in a fixed $k$-plane. Make a length $n$ binary string $\lambda$ with two zeros, in positions $n - k, n - j$, and let $S_\lambda$ denote the cohomology (or K-theory) class.

Then the above lets us compute

$$(S_{0101})^2 = S_{1001} + S_{0110} \quad \text{in } H^*(\text{Gr}(2, \mathbb{C}^4)) \quad \text{(or that minus } S_{1010}, \text{ in } K(\text{Gr}(2, \mathbb{C}^4)))$$
Cohomology and $K$-theory of Grassmannians.

To a length $n$ binary string $\lambda$ with $k$ zeroes, consider the Schubert cell

$$X^c_\lambda := \left\{ \begin{array}{c}
\text{row span }
\begin{bmatrix}
0 & 1 & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \\
\text{the } k \text{ pivot columns at } \lambda^\prime \text{s zeroes}
\end{array} \right\} \subseteq \text{Gr}(k, \mathbb{C}^n)$$

Using Gaussian elimination, we see these cells give a paving of $\text{Gr}(k, \mathbb{C}^n)$ by affine spaces, so their closures give bases $\{S_\lambda\}$ of cohomology and $K$-theory called Schubert classes. When we have a ring with basis $\{S_\lambda\}$, we want to understand the structure constants $c_{\lambda \mu}^\nu$ of its multiplication $S_\lambda S_\mu = \sum_{\nu} c_{\lambda \mu}^\nu S_\nu$.

**Theorem [Littlewood-Richardson 1934, made correct in 1970s]**
The $H^*$ structure constants count a set (of Young tableaux), so are $\geq 0$.

**Theorem [Kleiman 1973].** There’s a geometric reason for this, and it applies to other homogeneous spaces $G/P$ as well, but gives no formula. (Indeed, there is a Galois group obstruction to enumerating points of intersection [Harris 1979].) The corresponding results in $K$-theory are [Buch ’02], followed by [Brion ’02].
A first formula for the structure constants of $H^*_T(\text{Gr}(k, \mathbb{C}^n))$.

Theorem [K-Tao, ’03]. Glue these puzzle pieces (which may be rotated) into puzzles, which aren’t permitted 10-labels on the boundary.

Then in $H^*$, $c^\nu_{\lambda \mu}$ is the number of puzzles with boundary conditions $\lambda, \mu, \nu$ like so:

In fact our result is in torus-equivariant cohomology, with structure constants $c^\nu_{\lambda \mu}$ now in $H^*_T(\text{pt}) \cong \mathbb{Z}[y_1, \ldots, y_n]$:

$$(S_{0101})^2 = S_{1001} + S_{0110} + (y_2 - y_3)S_{0101}$$

The **equivariant piece** doesn’t break into triangles, can’t be rotated, and contributes a factor of $y_i - y_j$ according to its position.
Puzzles for 2-step and 3-step flag manifolds.

A d-step flag manifold $\text{Fl}(n_1, n_2, \ldots, n_d; \mathbb{C}^n)$ is the space of chains 
$\{0 \leq V^{n_1} \leq V^{n_2} \leq \ldots \leq V^{n_d} \leq \mathbb{C}^n\}$ of subspaces with a fixed list of dimensions, the $d = 1$ case being Grassmannians. This manifold too comes with a decomposition into Schubert cells, now indexed by strings in $\{0, 1, \ldots, d\}$ with multiplicities given by the differences $n_{i+1} - n_i$ (where $n_0 = 0, n_{d+1} = n$).

**Conjecture [K 1999], Theorem [Buch-Kresch-Purbhoo-Tamvakis ’16].**

The same puzzle count computes structure constants in $H^*(\text{Fl}(n_1, n_2; \mathbb{C}^n))$, requiring only these new puzzle pieces (& rotations):

![Puzzles](image)

It’s relatively easy to check that my rule gives the correct multiplication by generators. BKPT’s lengthy and delicate proof is that my rule is associative.

So, apparently one wants numbers 0, 1, 2 around the outside of the puzzle plus on the inside, “multinumbers” (XY) where all X > all Y? I found that the analogous 3-step multinumbers gave 23 labels and didn’t quite work.

**Corrected conjecture [Buch ’06], Theorem [K–Zinn-Justin ’17].**

The same puzzle count computes $d = 3$ structure constants, but one needs 27 labels, the ones I missed being $(3(21))(10), (32)((21)0), 3(((32)1)0), (3(2(10)))0$.

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Example. A 2-step puzzle in which all 8 labels appear.
A dual picture: scattering diagrams and a surprise.

The $n$ triangles on the bottom of a puzzle shape are different from the others: they can’t occur in an equivariant piece. Let’s pair up the non-bottom triangles into vertical rhombi. Now, let’s look at the graph-theory dual of an equivariant puzzle, an overlay of $n$ Ys.

This dual puzzle is worth $(y_1 - y_2)(y_2 - y_4)$:

If $V$ is the 3-d space with basis $\vec{0}, \vec{1}, \vec{10}$, then we can regard the options at a crossing as giving a matrix $R : V \otimes V \rightarrow V \otimes V$; at a trivalent vertex as a matrix $U : V \otimes V \rightarrow V^*$; and the puzzle formula as a matrix coefficient $V \otimes^{2n} \rightarrow (V^*) \otimes^n$.

That’s not quite right because of the $y_i - y_j$ coefficients; we need the tensor factors $V$ to “carry” these parameters in some sense, $(V, y_i)$.

**Observation [Zinn-Justin ’05].**

Rotating the nonrotatable equivariant pieces appropriately (!?), the equivariant puzzle $R$-matrix satisfies the **Yang-Baxter equation**:

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Where do solutions to Yang-Baxter (typically) come from?

Let $U_q(g[z^\pm])$ be the quantized loop algebra; it comes with many “evaluation representations” $(V_\delta, c \in \mathbb{C}^\times)$ taking $z \mapsto c$ then using the usual irrep $V_\delta$ of $g$.

Drinfel’d and Jimbo observed that $(V_\gamma, a) \otimes (V_\delta, b)$ is irreducible for generic $a/b$, but $\cong$ to $(V_\delta, b) \otimes (V_\gamma, a)$, and these isos are “$R$-matrices” (solutions to YBE).

**Theorem [K-ZJ ’17].** 1. The $d = 1$ puzzle $R$-matrix, acting on the $\otimes^2$ of the 3-space with basis $\{\vec{0}, \vec{1}, \vec{10}\}$, is a $q \to \infty$ limit of the $R$-matrix for $\mathfrak{sl}_3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$.

2. For the $d = 2$ case and its 8 edge labels $\vec{0}, \vec{1}, \vec{2}, \vec{10}, \vec{20}, \vec{21}, 2(\vec{10}), (2\vec{1})0$, we need a $q \to \infty$ limit of the $R$-matrix for $\mathfrak{d}_4 \otimes \text{spin}_+ \otimes \text{spin}_-$.

3. For the $d = 3$ case and its 27 edge labels, we need a $q \to \infty$ limit of the $R$-matrix for $\mathfrak{e}_6 \otimes \mathbb{C}^{27} \otimes \mathbb{C}^{27}$ (which one can find in the 1990s physics literature).

4. For the $d = 4$ case, the same technology led us to a 249-label rule based on $\mathfrak{e}_8 \otimes (\mathfrak{e}_8 \oplus \mathbb{C}) \otimes^2$, but alas it is nonpositive.

In each case, the Yang-Baxter equation (and similar “bootstrap” equation to deal with trivalent vertices) is used in a quick proof of the puzzle rule, and the nonzero matrix entries in the $q \to \infty$ limit tell us the valid puzzle pieces.

There was even no conjecture for $K$-theory in 2- or 3-step until 2017 (which arrived with our YBE-based proof, and in 3-step requires 151 new pieces).

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A more natural labeling of $d = 1$ puzzles.

The trivalent pieces are based on the map $\mathbb{C}^3 \otimes \mathbb{C}^3 \to \text{Alt}^2 \mathbb{C}^3$. Using the T-weights as labels (instead of 0, 1, 10) makes puzzles look more like pipe dreams:

In that old labeling system, the 10 label is forbidden on every boundary, but in the new one, the 2 is forbidden on NE, the 0 on NW, the 02 on South. In the old, we forbid the 10 – 10 – 10 triangle; in the new, the 02 label (everywhere).

To get back to the old labels (but don’t! they’re not as good), one first replaces each $ij$ with the unique missing label i.e. $\text{Alt}^2 \mathbb{C}^3 \cong (\mathbb{C}^3)^*$, then rotates the label system $0 \to 1 \to 2 \to 0$ once on the South side and twice on the NW side. Finally, write 2 as “10”.
Nakajima’s geometry of some $U_q(g[z^{\pm}])$ representations.

But why should such representations come up in studying $Fl(n_1, n_2, \ldots, n_d; \mathbb{C}^n)$? Given an oriented graph $(Q_0, Q_1)$, with some vertices declared “framed” and the others “gauged”, double it by adding a backwards arrow for every arrow. Attach a vector space $W_i$ to each framed vertex and $V_j$ to each gauged vertex.

Definition. A point in the quiver variety $M(Q_0, Q_1, W, V)$ is a choice of linear transformation for every edge, such that

- $\sum \pm$ (go out) $\circ$ (come back in) is zero at each gauged vertex;
- (“stability”) each $\vec{v}$ in each $V_i \setminus \partial$ can leak into some $W_j \setminus \partial$ via some path;
- all is considered up to $\prod_i GL(V_i)$ change-of-basis at the gauged vertices.

Let $M(Q_0, Q_1, W) := \bigsqcup_W M(Q_0, Q_1, W, V)$ be the quiver scheme.

Theorem [Nakajima ’01]. If $Q$ is ADE, then $U_q(its g[z^{\pm}]) \circ K(M(Q_0, Q_1, W))$.

Main example. $M \left( \begin{array}{c} n \\ \uparrow \\ n_d \leftarrow n_{d-1} \leftarrow \ldots \leftarrow n_1 \end{array} \right) \cong T^*Fl(n_1, \ldots, n_d; \mathbb{C}^n)$.

For this framing the $U_q(sl_{d+1}[z^{\pm}])$-action appears already in [Ginzburg-Vasserot 1993], and the rep is $K(M(Q_0, Q_1, n_1)) \cong (\mathbb{C}^{d+1})^n$, whose weight multiplicities are $(d+1)$-nomial coefficients, i.e. $= \dim K(T^*Fl(n_1, \ldots, n_d; \mathbb{C}^n))$. These slides are available at http://math.cornell.edu/~allenk/
Recognizing quiver varieties that are just $T^*\text{Fl}(n_1, \ldots, n_d; \mathbb{C}^n)$.

Obviously if the $V$ dimension vector is supported on a type $A$ subdiagram $S \subseteq Q$, and $W$ on a single vertex at one end of $S$, then by the last slide $\mathcal{M}(Q_0, Q_1, W, V) \cong T^*\text{Fl}(n_1, \ldots, n_d; \mathbb{C}^n)$. Say that these $(V, W)$ are of flag type. Nakajima defined “reflections” $\mathcal{M}(Q_0, Q_1, W, V, \theta) \cong \mathcal{M}(Q_0, Q_1, W, r_\alpha \cdot V, r_\alpha \cdot \theta)$ but they involve $\theta$-stability, in general more subtle than our “each $\vec{v} \in V_i$ leaks into some $W_j$” stability condition (which corresponds to $\forall \langle \theta_i, \alpha_j \rangle > 0$). If $\langle \theta_i, \alpha_j \rangle > 0$ for all $V_j > 0$, though, our naïve notion of stability is still correct.

The action of $r_\alpha \cdot V$ replaces the $\alpha$ label by the sum of the neighbors including the framed neighbor in $W$, minus the original label. In particular the new dimension is a linear combination of the original dimensions.

**Theorem [K-ZJ].** Assume $(Q_0, Q_1, W, V)$ is of flag type, and that the dimensions in $\pi \cdot V$ are nonnegative combinations of the dimensions in $V$. Then $\mathcal{M}(Q_0, Q_1, W, \pi \cdot V) \cong T^*\text{Fl}(n_1, \ldots, n_d; \mathbb{C}^n)$, steps coming from $\text{dim } V$.

Some $D_4$ examples.

\[
\begin{pmatrix} 0 & j & k \\ 0 & j & k \end{pmatrix} \rightarrow \begin{pmatrix} j & j & k \\ j & j & k \end{pmatrix} \rightarrow \begin{pmatrix} j & j+k & k \\ j & j+k & k \end{pmatrix}
\]

\[
\begin{pmatrix} n \\ n \end{pmatrix} \rightarrow \begin{pmatrix} n \\ n \end{pmatrix} \rightarrow \begin{pmatrix} n \\ n \end{pmatrix}
\]

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Some Lagrangian relations of quiver varieties.

Recall that we decided that the puzzle labels should be $0^k, 1^{n-k}$ on NE but $1^k, 2^{n-k}$ on NW, suggesting we work with “2-step” $\text{Fl}(k, n; \mathbb{C}^n)$ and $\text{Fl}(0, k; \mathbb{C}^n)$. On $\mathbb{C}^n \oplus \mathbb{C}^n$ let’s put a $\mathbb{C}^\times$-action with weights 0, 1, extending to an action on $M\left(\begin{array}{c} n+n \\ n+k \\ k \end{array}\right)$; then $M\left(\begin{array}{c} n \\ k \\ 0 \end{array}\right) \times M\left(\begin{array}{c} n \\ n \\ k \end{array}\right)$ is a fixed-point component. Let $\text{attr}$ be the (closed!) attracting set, the Morse/Białynicki-Birula stratum.

Now let $\Phi^{-1}_N(1) := \{\text{the composite } (\mathbb{C}^n \oplus 0) \searrow \mathbb{C}^{n+k} \nearrow (0 \oplus \mathbb{C}^n) \text{ is the identity}\}$. Points (reps) in that set enjoy splittings of $\mathbb{C}^{n+k}$, plus coordinates on the $\mathbb{C}^n$.

**Imprecisely stated theorem [K-ZJ].** The Lagrangian relations

$$M\left(\begin{array}{c} n \\ k \\ 0 \end{array}\right) \times M\left(\begin{array}{c} n \\ n \\ k \end{array}\right) \xleftarrow{\text{attr}} M\left(\begin{array}{c} n+n \\ n+k \\ k \end{array}\right) \xleftarrow{\Phi^{-1}_N(1)} M\left(\begin{array}{c} n \\ n \\ k \end{array}\right)$$

induce the usual multiplication map on $H^*_\mathbb{T \times \mathbb{C}^\times}(T^* \text{Gr}(k, \mathbb{C}^n))$, up to a scale, and by following the natural (analogues of Schubert) bases (and taking $q$, or really $\hbar$, to $\infty$) we recover Grassmannian puzzles.

Changing the left $k$ to $j$ gives $H^*(\text{Gr}(j, \mathbb{C}^n)) \otimes H^*(\text{Gr}(k, \mathbb{C}^n)) \rightarrow H^*(\text{Fl}(j, k; \mathbb{C}^n))$, i.e. all this time the 1-step puzzle pieces were already enough to do some 2-step!

These slides are available at http://math.cornell.edu/~allenk/
Quiver varieties that recover $d = 2, 3$ puzzles.

Each of the below reflects to a flag type quiver variety, which is fun to verify.

$d = 2$:
\[
\begin{pmatrix}
\text{n} & j & 0 \\
k & 0 & 0 \\
\end{pmatrix}
\times
\begin{pmatrix}
n & n+k & n+j \\
k & & \\
\end{pmatrix}
\text{sum, then split using}
\begin{pmatrix}
m \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
m \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
k & k+j & j \\
k & & \\
m & & \\
\end{pmatrix}
\text{remember me from slide 10?}
\]

$d = 3$:
\[
\begin{pmatrix}
\text{n} & l & k & j & 0 & 0 \\
k & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\times
\begin{pmatrix}
\text{n} & 2n & 2n+l & 2n+l+k & n+l+j & l \\
2n & 2n & 2n & 2n & 2n & 2n \\
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
\text{n} & 2n & 2n+l & 2n+l+k & n+l+j & l \\
2n & 2n & 2n & 2n & 2n & 2n \\
\end{pmatrix}
\]
\[
\leftrightarrow
\begin{pmatrix}
l & l+k & l+k+j & l+j & l \\
\end{pmatrix}
\]

this Lagrangian relation involves two matrix equations

We know some $E_8$ quiver varieties giving $d = 4$, but the corresponding reps $\mathfrak{e}_8 \oplus \mathbb{C}$ are not multiplicity-free, and don’t lead to a positive rule. (It’s a mostly positive rule, and surely the most efficient known, but definitely not positive.)
Multiplying Segre-Schwartz-MacPherson classes.

If we keep $q$ around, instead of taking it to $\infty$, we get classes in $K_C \times (\text{T}^*\text{Fl}(j, k; \mathbb{C}^n))$ associated to certain conical-Lagrangian-supported sheaves. Puzzles then compute the products of a related set: those classes, but divided by the class of the zero section (also Lagrangian). These puzzles also compute (in the $K \rightarrow H^*$ limit) the \textit{comultiplication} of Chern-Schwarz-MacPherson classes.

The Grassmannian rule has puzzle pieces for \textit{all} nonzero matrix entries of $\mathbb{C}^3 \otimes \mathbb{C}^3 \rightarrow \text{Alt}^2 \mathbb{C}^3$; unlike as in ordinary puzzles, this rule doesn’t forbid the 02 label (those entries are suppressed only in the $q \rightarrow 0, K \rightarrow H^*$ limit).

\textbf{Theorem [K-ZJ].} The CSM result lets one compute compactly supported Euler characteristics of intersections of generically translated Bruhat cells:

$$\chi_c \left( \bigcap_{i=1}^{3} (g_i \cdot X_{\lambda_i}^o) \right) = (-1)^{k(n-k)} \sum_{i=1}^{3} \ell(\lambda_i) \# \left\{ \text{puzzles now including 02 labels} \right\}$$

\textit{Example.} Intersect three open Bruhat cells on $\mathbb{CP}^1$ transversely, resulting in $\mathbb{CP}^1 \setminus \{3 \text{ points}\}$. That has $\chi_c = 2 - 3(1) = -1^{(2-1)}$, and indeed there is one puzzle, using the 02 label in the interior.
The newest Schubert calculus: separated descents.

**Theorem [K-ZJ].** Consider the puzzle pieces at right, and their $180^\circ$ rotations. Make size $n$ puzzles with $1, \ldots, k$ and $n - k$ blanks on NE side, $k + 1, \ldots, n$ and $k$ blanks on NW side. Then these puzzles compute pullbacks of classes along $\text{Fl}(n_1, \ldots, n_d; \mathbb{C}^n) \hookrightarrow \text{Fl}(n_1, \ldots, n_k; \mathbb{C}^n) \times \text{Fl}(n_k, \ldots, n_d; \mathbb{C}^n)$ and with two more pieces (next slide) we get the $K_T$-version.

[Kogan ’01], the previous state-of-the-art for general $H^*(\text{Fl}(\mathbb{C}^n))$ calculations (extended to $K$-theory in [K-Yong ’04]), assumed that one of the two factors was a Grassmannian. (Also this rule was algorithmic, and nonequivariant.)

**“Proof”**. Same recipe as slide 11, using the Lagrangian relations

$$\mathcal{M}\left(\begin{array}{cccc} n & n \ldots n & n_k \ldots n_1 \end{array}\right) \times \mathcal{M}\left(\begin{array}{ccc} n & n_1 \ldots n_k & 0 \ldots 0 \end{array}\right) \quad \text{attr closed, by greedy splitting}$$

$$\mathcal{M}\left(\begin{array}{cccc} n+n & n \ldots n \end{array}\right)$$

$\Phi_N^{-1}(1) \mapsto \mathcal{M}\left(\begin{array}{cccc} n & n & n \ldots n \end{array}\right)$

$\mathcal{M}\left(\begin{array}{cccc} n & n & n \ldots n \end{array}\right) \sim T^*\text{Fl}(\mathbb{C}^n)$

These slides are available at http://math.cornell.edu/~allenk/
A sample separated-descents puzzle, and, the equivariant and $K$-theoretic (and dual-$K$-theoretic) pieces.