Schubert calculus puzzles from quiver varieties

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Flags, Galleries, and Reflection Groups, Sydney 2019

Abstract

In 2006 Paul Zinn-Justin observed that our puzzle rule [K-Tao '03] for equivariant Schubert calculus on Grassmannians was based on an "R-matrix", a solution to the Yang-Baxter equation. In 2017 Zinn-Justin and I extended this to discover and prove puzzle rules for K-theory of 2- and 3-step flag manifolds.

[Maulik and Okounkov '12] trace R-matrices to Nakajima quiver varieties (whose definition I'll recall), and I'll explain how our puzzles can be seen directly from the quiver varieties. (In fact, the puzzles for a quiver variety extension are more symmetric!) We give a rule to recognize when a general-looking quiver variety is just T^* of a partial flag variety.

Then I'll show a further extension, which was most easily discovered via the quiver variety interpretation, computing pullbacks in K_T along $Fl(\mathbb{C}^n) \hookrightarrow Fl(k, k+1, \ldots, n; \mathbb{C}^n) \times Fl(1, 2, \ldots, k; \mathbb{C}^n)$.

These slides are available at `http://math.cornell.edu/~allenk/`

An intersection theory problem.

Let L_1, L_2 be two different, but crossing, lines in 3-space.
Let Y_1, Y_2 be the set of lines touching L_1, L_2 respectively. Then

$$Y_1 \cap Y_2 \quad = \quad \{\text{lines in the } L_1L_2 \text{ plane}\} \bigcup_{\{\text{lines doing both}\}} \{\text{lines through } L_1 \cap L_2\}$$

Let $\mathrm{Gr}(1, \mathbb{P}^3) \cong \mathrm{Gr}(2, \mathbb{C}^4)$ be the **Grassmannian** of lines in projective 3-space. Although $Y_1 \neq Y_2$ as sets, they are homologous in $\mathrm{Gr}(2, \mathbb{C}^4)$, so define the same element "S_{0101}" in cohomology (or K-theory).

More generally, consider lines in $\mathbb{P}^{n-1}$ that touch a fixed j-plane and are contained in a fixed k-plane. Make a length n binary string λ with two zeros, in positions $n-k, n-j$, and let S_λ denote the cohomology (or K-theory) class.

Then the above lets us compute

$$(S_{0101})^2 = S_{1001} + S_{0110} \quad \text{in } H^*(\mathrm{Gr}(2, \mathbb{C}^4)) \quad \left(\text{or that minus } S_{1010}, \text{ in } K(\mathrm{Gr}(2, \mathbb{C}^4))\right)$$

Cohomology and K-theory of Grassmannians.

To a length n binary string λ with k zeroes, consider the **Schubert cell**

$$X^\circ_\lambda := \left\{ \begin{matrix} \text{row} \\ \text{span} \end{matrix} \begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \atop \text{the } k \text{ pivot columns at } \lambda\text{'s zeroes} \right\} \qquad \subseteq Gr(k, \mathbb{C}^n)$$

Using Gaussian elimination, we see these cells give a paving of $Gr(k, \mathbb{C}^n)$ by affine spaces, so their closures give bases $\{S_\lambda\}$ of cohomology and K-theory called **Schubert classes**. When we have a ring with basis $\{S_\lambda\}$, we want to understand the structure constants $c^\nu_{\lambda\mu}$ of its multiplication $S_\lambda S_\mu = \sum_\nu c^\nu_{\lambda\mu} S_\nu$.

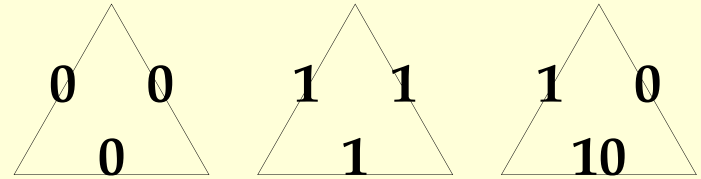
Theorem [Littlewood-Richardson 1934, made correct in 1970s]
The H^* structure constants count a set (of Young tableaux), so are ≥ 0.

Theorem [Kleiman 1973]. There's a geometric reason for this, and it applies to other homogeneous spaces G/P as well, but gives no formula. (Indeed, there is a Galois group *obstruction* to enumerating points of intersection [Harris 1979].)

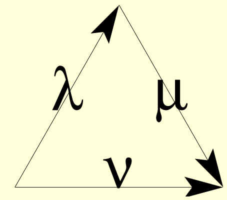
The corresponding results in K-theory are [Buch '02], followed by [Brion '02].

A first formula for the structure constants of $H_T^*(Gr(k, \mathbb{C}^n))$.

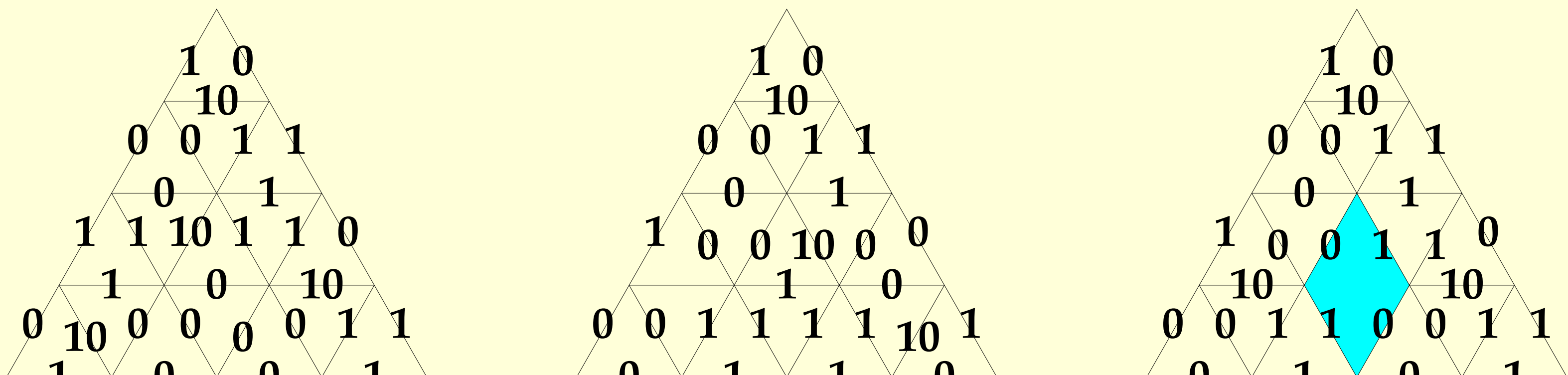
Theorem [K-Tao, '03]. Glue these **puzzle pieces** (which may be rotated) into **puzzles**, which aren't permitted 10-labels on the boundary.



Then in H^*, $c_{\lambda\mu}^{\nu}$ is the number of puzzles with boundary conditions λ, μ, ν like so:



In fact our result is in *torus-equivariant* cohomology, with structure constants $c_{\lambda\mu}^{\nu}$ now in $H_T^*(pt) \cong \mathbb{Z}[y_1, \ldots, y_n]$:



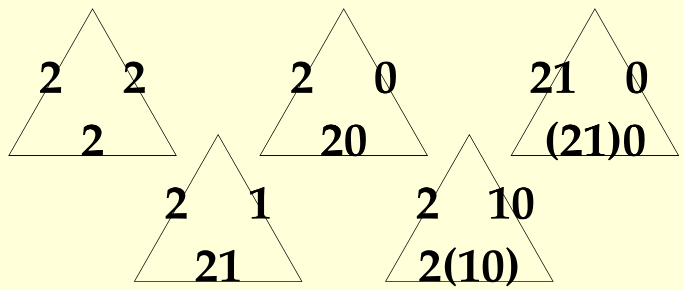
$$(S_{0101})^2 = S_{1001} + S_{0110} + (y_2 - y_3) S_{0101}$$

The **equivariant piece** doesn't break into triangles, *can't be rotated,* and contributes a factor of $y_i - y_j$ according to its position.

Puzzles for 2-step and 3-step flag manifolds.

A **d-step flag manifold** $Fl(n_1, n_2, \ldots, n_d; \mathbb{C}^n)$ is the space of chains $\{0 \leq V^{n_1} \leq V^{n_2} \leq \ldots \leq V^{n_d} \leq \mathbb{C}^n\}$ of subspaces with a fixed list of dimensions, the $d = 1$ case being Grassmannians. This manifold too comes with a decomposition into Schubert cells, now indexed by strings in $\{0, 1, \ldots, d\}$ with multiplicities given by the differences $n_{i+1} - n_i$ (where $n_0 = 0, n_{d+1} = n$).

Conjecture [K 1999], Theorem [Buch-Kresch-Purbhoo-Tamvakis '16].
The same puzzle count computes structure constants in $H^*(Fl(n_1, n_2; \mathbb{C}^n))$, requiring only these new puzzle pieces (& rotations):

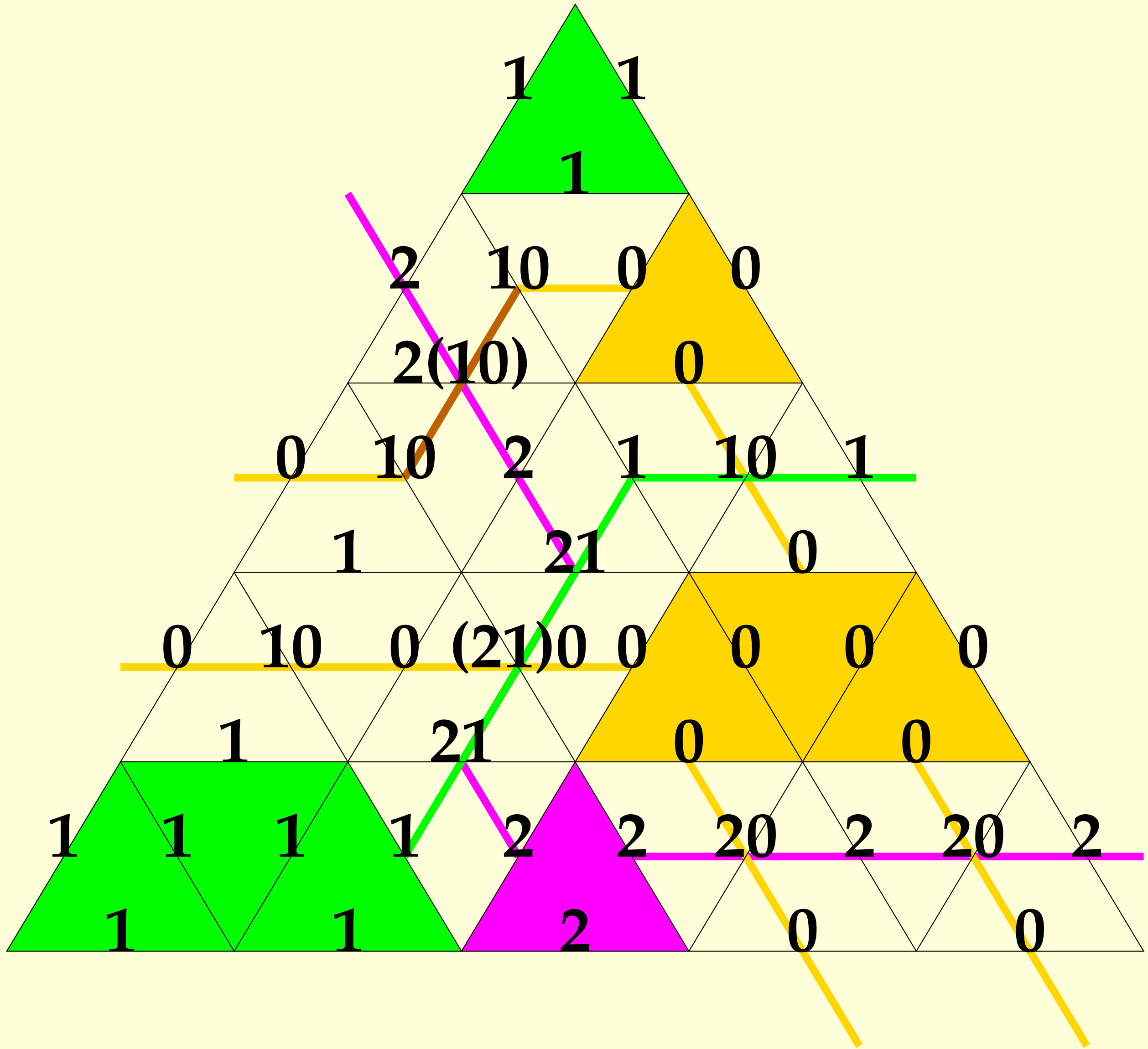


It's relatively easy to check that my rule gives the correct multiplication by generators. BKPT's lengthy and delicate proof is that my rule is *associative*.

So, apparently one wants numbers $0, 1, 2$ around the outside of the puzzle plus on the inside, "multinumbers" (XY) where all $X >$ all Y? I found that the analogous 3-step multinumbers gave 23 labels and didn't quite work.

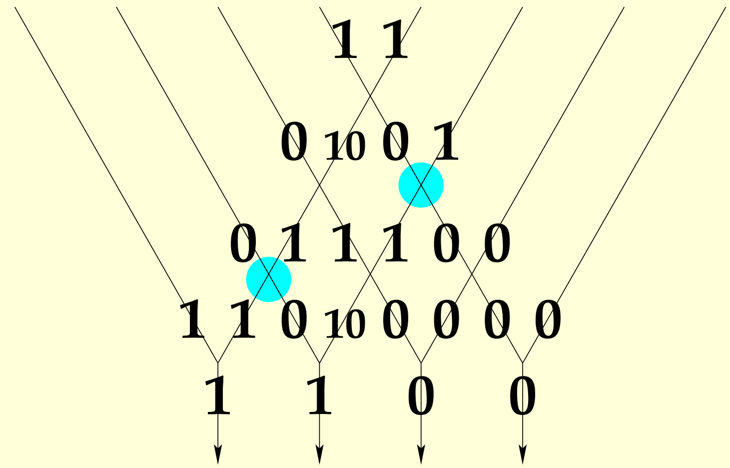
Corrected conjecture [Buch '06], Theorem [K–Zinn-Justin '17].
The same puzzle count computes $d = 3$ structure constants, but one needs 27 labels, the ones I missed being $(3(21))(10), (32)((21)0), 3(((32)1)0), (3(2(10)))0$.

Example. A 2-step puzzle in which all 8 labels appear.



A dual picture: scattering diagrams and a surprise.

The n triangles on the bottom of a puzzle shape are different from the others: they can't occur in an equivariant piece. Let's pair up the non-bottom triangles into vertical rhombi.
Now, let's look at the graph-theory dual of an equivariant puzzle, an overlay of n Ys.

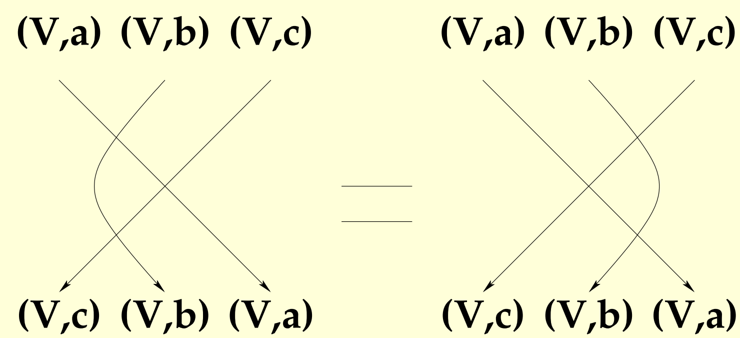


This dual puzzle is worth $(y_1 - y_2)(y_2 - y_4)$:

If V is the 3-d space with basis $\vec{0}, \vec{1}, \vec{10}$, then we can regard the options at a crossing as giving a matrix $R: V \otimes V \to V \otimes V$; at a trivalent vertex as a matrix $U: V \otimes V \to V^*$; and the puzzle formula as a matrix coefficient $V^{\otimes 2n} \to (V^*)^{\otimes n}$.

That's not quite right because of the $y_i - y_j$ coefficients; we need the tensor factors V to "carry" these parameters in some sense, (V, y_i).

Observation [Zinn-Justin '05].
Rotating the nonrotatable equivariant pieces appropriately (!?), the equivariant puzzle R-matrix satisfies the **Yang-Baxter equation:**



Where do solutions to Yang-Baxter (typically) come from?

Let $U_q(\mathfrak{g}[z^\pm])$ be the **quantized loop algebra**; it comes with many "evaluation representations" $(V_\delta, c \in \mathbb{C}^\times)$ taking $z \mapsto c$ then using the usual irrep V_δ of $\mathfrak{g}$.

Drinfel′d and Jimbo observed that $(V_\gamma, a) \otimes (V_\delta, b)$ is irreducible for generic a/b, but $\cong$ to $(V_\delta, b) \otimes (V_\gamma, a)$, and these isos are "R-matrices" (solutions to YBE).

Theorem [K-ZJ '17]. 1. The $d = 1$ puzzle R-matrix, acting on the $\otimes^2$ of the 3-space with basis $\{\vec{0}, \vec{1}, \vec{10}\}$, is a $q \to \infty$ limit of the R-matrix for $\mathfrak{sl}_3 \circlearrowright \mathbb{C}^3 \otimes \mathbb{C}^3$.

2. For the $d = 2$ case and its 8 edge labels $\vec{0}, \vec{1}, \vec{2}, \vec{10}, \vec{20}, \vec{21}, 2(\vec{10}), (\vec{21})0$, we need a $q \to \infty$ limit of the R-matrix for $\mathfrak{d}_4 \circlearrowright \mathtt{spin}_+ \otimes \mathtt{spin}_-$.

3. For the $d = 3$ case and its 27 edge labels, we need a $q \to \infty$ limit of the R-matrix for $\mathfrak{e}_6 \circlearrowright \mathbb{C}^{27} \otimes \mathbb{C}^{27}$ (which one can find in the 1990s physics literature).

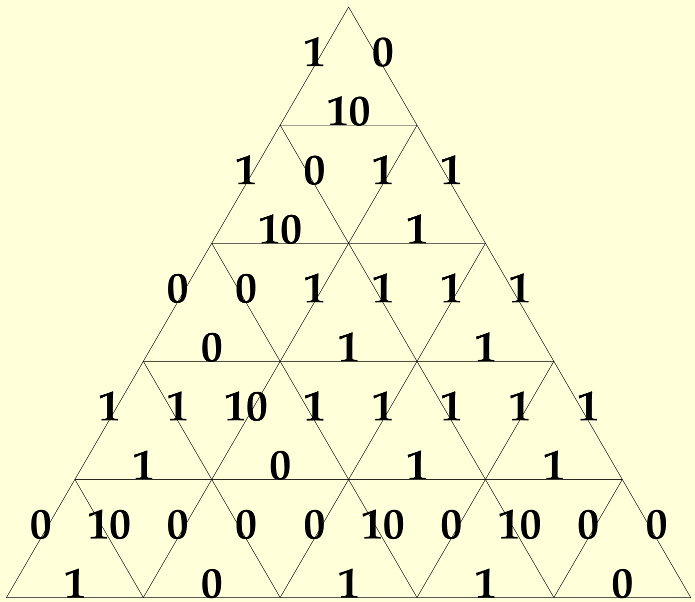
4. For the $d = 4$ case, the same technology led us to a 249-label rule based on $\mathfrak{e}_8 \circlearrowright (\mathfrak{e}_8 \oplus \mathbb{C})^{\otimes 2}$, but alas it is ***nonpositive***. ☹

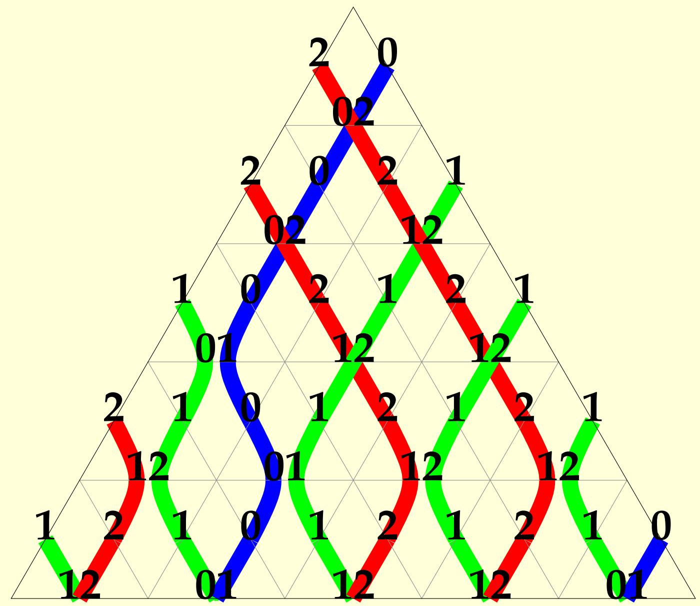
In each case, the Yang-Baxter equation (and similar "bootstrap" equation to deal with trivalent vertices) is used in a quick proof of the puzzle rule, and the nonzero matrix entries in the $q \to \infty$ limit tell us the valid puzzle pieces.

There was even no *conjecture* for K-theory in 2- or 3-step until 2017 (which arrived with our YBE-based proof, and in 3-step requires 151 new pieces).

A more natural labeling of $d = 1$ puzzles.

The trivalent pieces are based on the map $\mathbb{C}^3 \otimes \mathbb{C}^3 \to \mathrm{Alt}^2\mathbb{C}^3$. Using the T-weights as labels (instead of $0, 1, 10$) makes puzzles look more like pipe dreams:





In that old labeling system, the 10 label is forbidden on every boundary, but in the new one, the 2 is forbidden on NE, the 0 on NW, the 02 on South. In the old, we forbid the $10 - 10 - 10$ triangle; in the new, the 02 label (everywhere).

To get back to the old labels (but don't! they're not as good), one first replaces each ij with the unique *missing* label i.e. $\mathrm{Alt}^2\mathbb{C}^3 \cong (\mathbb{C}^3)^*$, then rotates the label system $0 \to 1 \to 2 \to 0$ once on the South side and twice on the NW side. Finally, write 2 as "10".

Nakajima's geometry of some $U_q(\mathfrak{g}[z^\pm])$ representations.

But why *should* such representations come up in studying $Fl(n_1, n_2, \ldots, n_d; \mathbb{C}^n)$?

Given an oriented graph (Q_0, Q_1), with some vertices declared "$\boxed{\text{framed}}$" and the others "gauged", double it by adding a backwards arrow for every arrow. Attach a vector space $\boxed{W_i}$ to each $\boxed{\text{framed}}$ vertex and V_j to each gauged vertex.

Definition. A point in the **quiver variety** $\mathcal{M}(Q_0, Q_1, \boxed{W}, V)$ is a choice of linear transformation for every edge, such that

- $\sum \pm$ (go out) $\circ$ (come back in) is zero at each gauged vertex;
- ("stability") each $\vec{v}$ in each $V_i \setminus \vec{0}$ can leak into some $\boxed{W_j \setminus \vec{0}}$ via *some* path;
- all is considered up to $\prod_i GL(V_i)$ change-of-basis at the gauged vertices.

Let $\mathcal{M}(Q_0, Q_1, \boxed{W}) := \coprod_{\boxed{W}} \mathcal{M}(Q_0, Q_1, \boxed{W}, V)$ be the **quiver scheme**.

Theorem [Nakajima '01]. If Q is ADE, then $U_q(\text{its } \mathfrak{g}[z^\pm]) \circlearrowright K(\mathcal{M}(Q_0, Q_1, W))$.

Main example.
$$\mathcal{M}\begin{pmatrix} \boxed{n} & & & & \\ \uparrow & & & & \\ n_d & \leftarrow & n_{d-1} & \leftarrow \ldots \leftarrow & n_1 \end{pmatrix} \cong T^*Fl(n_1, \ldots, n_d;\ \mathbb{C}^n).$$

For this framing the $U_q(\mathfrak{sl}_{d+1}[z^\pm])$-action appears already in [Ginzburg-Vasserot 1993], and the rep is $K(\mathcal{M}(Q_0, Q_1, n\omega_1)) \cong (\mathbb{C}^{d+1})^{\otimes n}$, whose weight multiplicities are $(d+1)$-nomial coefficients, i.e. $= \dim K(T^*Fl(n_1, \ldots, n_d; \mathbb{C}^n))$.

Recognizing quiver varieties that are just $T^*Fl(n_1, \ldots, n_d; \mathbb{C}^n)$.

Obviously if the V dimension vector is supported on a type A subdiagram $S \subseteq Q$, and W on a single $\boxed{\text{vertex}}$ at one end of S, then by the last slide $\mathcal{M}(Q_0, Q_1, \boxed{W}, V) \cong T^*Fl(n_1, \ldots, n_d; \mathbb{C}^n)$. Say that these (V, W) are of **flag type**.

Nakajima defined "reflections" $\mathcal{M}(Q_0, Q_1, \boxed{W}, V, \theta) \cong \mathcal{M}(Q_0, Q_1, \boxed{W}, r_\alpha \cdot V, r_\alpha \cdot \theta)$ but they involve θ-stability, in general more subtle than our "each $\vec{v} \in V_i$ leaks into some $\boxed{W_j}$" stability condition (which corresponds to $\forall \langle \theta_i, \alpha_j \rangle > 0$). If $\langle \theta_i, \alpha_j \rangle > 0$ for all $V_j > 0$, though, our naïve notion of stability is still correct.

The action of $r_\alpha \cdot V$ replaces the α label by the sum of the neighbors **including the $\boxed{\text{framed}}$ neighbor in $\boxed{W}$**, minus the original label. In particular the new dimension is a linear combination of the original dimensions.

Theorem [K-ZJ]. Assume $(Q_0, Q_1, \boxed{W}, V)$ is of flag type, and that the dimensions in $\pi \cdot V$ are nonnegative combinations of the dimensions in V. Then $\mathcal{M}(Q_0, Q_1, \boxed{W}, \pi \cdot V) \cong T^*Fl(n_1, \ldots, n_d; \mathbb{C}^n)$, steps coming from $\dim V$.

Some D_4 *examples.*

$$\begin{pmatrix} & & \boxed{n} \\ 0 & j & k \\ & 0 & \end{pmatrix} \to \begin{pmatrix} & & \boxed{n} \\ \mathbf{j} & j & k \\ & \mathbf{j} & \end{pmatrix} \to \begin{pmatrix} & & \boxed{n} \\ j & \mathbf{j}+\mathbf{k} & k \\ & j & \end{pmatrix}$$

$$\to \begin{pmatrix} & & \boxed{n} \\ \mathbf{k} & j+k & \mathbf{n}+\mathbf{j} \\ & \mathbf{k} & \end{pmatrix} \to \begin{pmatrix} & & \boxed{n} \\ k & \mathbf{n}+\mathbf{k} & n+j \\ & k & \end{pmatrix} \to \begin{pmatrix} & & \boxed{n} \\ \mathbf{n} & n+k & n+j \\ & k & \end{pmatrix}$$

Some Lagrangian relations of quiver varieties.

Recall that we decided that the puzzle labels should be $0^k, 1^{n-k}$ on NE but $1^k, 2^{n-k}$ on NW, suggesting we work with "2-step" $Fl(k, n;\ \mathbb{C}^n)$ and $Fl(0, k;\ \mathbb{C}^n)$.

On $\mathbb{C}^n \oplus \mathbb{C}^n$ let's put a $\mathbb{C}^\times$-action with weights $0, 1$, extending to an action on $\mathcal{M}\begin{pmatrix} \boxed{n+n} & \\ n+k & k \end{pmatrix}$; then $\mathcal{M}\begin{pmatrix} \boxed{n} & \\ k & 0 \end{pmatrix} \times \mathcal{M}\begin{pmatrix} \boxed{n} & \\ n & k \end{pmatrix}$ is a fixed-point component. Let attr be the **(closed!)** attracting set, the Morse/Białynicki-Birula stratum.

Now let $\Phi_N^{-1}(\mathbf{1}) := \{$the composite $(\mathbb{C}^n \oplus 0) \searrow \mathbb{C}^{n+k} \nearrow (0 \oplus \mathbb{C}^n)$ is the identity$\}$. Points (reps) in that set enjoy splittings of $\mathbb{C}^{n+k}$, plus coordinates on the $\mathbb{C}^n$.

Imprecisely stated theorem [K-ZJ]. The Lagrangian relations

$$\mathcal{M}\begin{pmatrix} \boxed{n} & \\ k & 0 \end{pmatrix} \times \mathcal{M}\begin{pmatrix} \boxed{n} & \\ n & k \end{pmatrix} \overset{\mathrm{attr}}{\longleftrightarrow} \mathcal{M}\begin{pmatrix} \boxed{n+n} & \\ n+k & k \end{pmatrix} \overset{\Phi_N^{-1}(\mathbf{1})}{\longleftrightarrow} \mathcal{M}\begin{pmatrix} & \boxed{n} \\ k & k \end{pmatrix}$$

induce the usual multiplication map on $H^*_{T\times\mathbb{C}^\times}(T^*Gr(k, \mathbb{C}^n))$, up to a scale, and by following the natural (analogues of Schubert) bases (and taking q, or really $\hbar$, to ∞) we recover Grassmannian puzzles.

Changing the left k to j gives $H^*(Gr(j, \mathbb{C}^n)) \otimes H^*(Gr(k, \mathbb{C}^n)) \to H^*(Fl(j, k;\ \mathbb{C}^n))$, i.e. all this time the 1-step puzzle pieces were already enough to do some 2-step!

Quiver varieties that recover $d = 2, 3$ puzzles.

Each of the below reflects to a flag type quiver variety, which is fun to verify.

$$d = 2: \quad \begin{pmatrix} \boxed{n} & & \\ k & j & 0 \\ & 0 & \end{pmatrix} \times \begin{pmatrix} & & \boxed{n} \\ n & n+k & n+j \\ & k & \\ \end{pmatrix} \xleftrightarrow{\text{sum, then split using } \boxed{n} \xrightarrow{1} \boxed{n}} \begin{pmatrix} k & k+j & j \\ & k & \\ & \boxed{n} & \end{pmatrix}$$

remember me from slide 10?

$$d = 3: \quad \begin{pmatrix} \boxed{n} & & & & \\ l & k & j & 0 & 0 \\ & & 0 & & \end{pmatrix} \times \begin{pmatrix} \boxed{n} & & & & \\ 2n & 2n+l & 2n+l+k & n+l+j & l \\ & & n+k & & \end{pmatrix}$$

this Lagrangian relation involves *two* matrix equations

$$\leftrightarrow \begin{pmatrix} & & & & \boxed{n} \\ l & l+k & l+k+j & l+j & l \\ & & k & & \end{pmatrix}$$

We know some E_8 quiver varieties giving $d = 4$, but the corresponding reps $\mathfrak{e}_8 \oplus \mathbb{C}$ are not multiplicity-free, and don't lead to a positive rule.
(It's a *mostly* positive rule, and surely the most efficient known, but definitely not positive.)

Multiplying Segre-Schwartz-MacPherson classes.

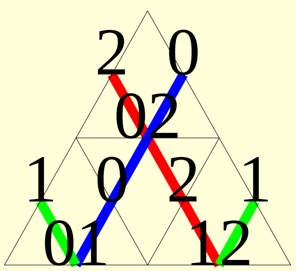
If we keep q around, instead of taking it to ∞, we get classes in $K_{\mathbb{C}^\times}(T^*Fl(j,k;\mathbb{C}^n))$ associated to certain conical-Lagrangian-supported sheaves. Puzzles then compute the products of a related set: those classes, but divided by the class of the zero section (also Lagrangian). These puzzles also compute (in the $K \dashrightarrow H^*$ limit) the *comultiplication* of Chern-Schwarz-MacPherson classes.

The Grassmannian rule has puzzle pieces for *all* nonzero matrix entries of $\mathbb{C}^3 \otimes \mathbb{C}^3 \to \mathrm{Alt}^2\mathbb{C}^3$; unlike as in ordinary puzzles, this rule doesn't forbid the 02 label (those entries are suppressed only in the $q \to 0$, $K \dashrightarrow H^*$ limit).

Theorem [K-ZJ]. The CSM result lets one compute compactly supported Euler characteristics of intersections of generically translated Bruhat cells:

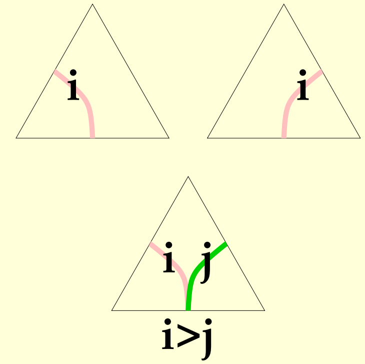
$$\chi_c\left(\bigcap_{i=1}^{3}(g_i \cdot X^\circ_{\lambda_i})\right) = (-1)^{k(n-k)-\sum_{i=1}^{3}\ell(\lambda_i)}\ \#\left\{\text{puzzles now including } 02 \text{ labels}\right\}$$

Example. Intersect three open Bruhat cells on $\mathbb{CP}^1$ transversely, resulting in $\mathbb{CP}^1 \setminus \{3 \text{ points}\}$. That has $\chi_c = 2 - 3(1) = -1^{1(2-1)}$, and indeed there is one puzzle, using the 02 label in the interior.



The newest Schubert calculus: separated descents.

Theorem [K-ZJ]. Consider the puzzle pieces at right, and their 180° rotations. Make size n puzzles with $1,\ldots,k$ and $n-k$ blanks on NE side, $k+1,\ldots,n$ and k blanks on NW side. Then these puzzles compute pullbacks of classes along $Fl(n_1,\ldots\ldots,n_d;\mathbb{C}^n) \hookrightarrow Fl(n_1,\ldots,n_k;\mathbb{C}^n) \times Fl(n_k,\ldots,n_d;\mathbb{C}^n)$ and with two more pieces (next slide) we get the K_T-version.



[Kogan '01], the previous state-of-the-art for general $H^*(Fl(\mathbb{C}^n))$ calculations (extended to K-theory in [K-Yong '04]), assumed that one of the two factors was a Grassmannian. (Also this rule was algorithmic, and nonequivariant.)

"Proof". Same recipe as slide 11, using the Lagrangian relations

$$\mathcal{M}\begin{pmatrix} \boxed{n} & & \\ n & n\ldots n & n_k\ldots n_1 \end{pmatrix} \times \mathcal{M}\begin{pmatrix} \boxed{n} & & \\ n_d & n_{d-1}\ldots n_k & 0\ldots 0 \end{pmatrix} \quad \text{attr closed, by greedy splitting}$$

$$\overset{attr}{\longleftrightarrow} \mathcal{M}\begin{pmatrix} \boxed{n+n} & & & \\ n+n_d & n+n_{d-1} & n+n_{d-2}\ldots n+n_k & n_k\ldots n_1 \end{pmatrix} \quad \text{of this}$$

$$\overset{\Phi_N^{-1}(1)}{\longleftrightarrow} \mathcal{M}\begin{pmatrix} & \boxed{n} & & \\ n_d & n+n_{d-1} & n+n_{d-2}\ldots n+n_k & n_k\ldots n_1 \end{pmatrix} \cong T^*Fl(\mathbb{C}^n)$$

A sample separated-descents puzzle, and, the equivariant and K-theoretic (and dual-K-theoretic) pieces.