# Statement of Research

**Andrew Marshall**

Papers: Linear Configurations of Complete Graphs $K_4$ and $K_5$ in $\mathbb{R}^3$, and Higher Dimensional Analogs. **arXiv 1450:1805** Submitted Jan 2015

My current research interest is in the geometry and topology of configuration spaces. Configuration spaces are spaces of positions of one object in another, and thus they naturally arise ubiquitously in mathematics as well as in physics, (e.g., in the $n$-body problem and symplectic geometry), in chemistry, (e.g., in the form of polymer folding) and in engineering, (e.g., in robotic motion control). In a suitable category, a configuration space is the space of images of embeddings of an object $X$ in an object $Y$. Said in other words, $\mathrm{Aut(X)}$, the group of automorphisms of $X$, acts on $\mathrm{Emb(X,Y)}$ by precomposition, effectively reparametrizing the embedding. Quotienting by this action gives the configuration space

$$C(X,Y) = \mathrm{Emb(X,Y)/Aut(X)}$$

(in the cases $\mathrm{Emb(X,Y)}$ is a topological space).

The case where $X$ is $\mathbf{n}$, the set of $n$ points, and $Y$ is $\mathbb{R}^2$ results in a configuration space which is an Eilenberg-MacLane space with fundamental group $B_n$, the braid group on $n$ strands. This group was introduced in the work of Hurwitz and studied extensively by Artin. It continues to be important to a number of areas of mathematics, including group theory, cryptology, celestial mechanics, and knot theory. The complement of a configuration has fundamental group $F_n$, the free group on $n$ generators. A loop in the configuration space acts on loops in the complement of the base configuration by dragging them along, thus $B_n$ acts on $F_n$. This action realizes $B_n$ as a subgroup of $\mathrm{Aut(F_n)}$.

Generalizations both where $Y$ is a surface, where it is a graph, and where it is a lens space have been studied in detail, and has produced many interesting results. The generalization where $Y$ is $\mathbb{R}^3$ and $X$ is a number of disjoint unknotted and unlinked $C^\infty$ embedded circles was introduced in the thesis of Dahm and studied in a 2002 paper of Brendle and Hatcher where

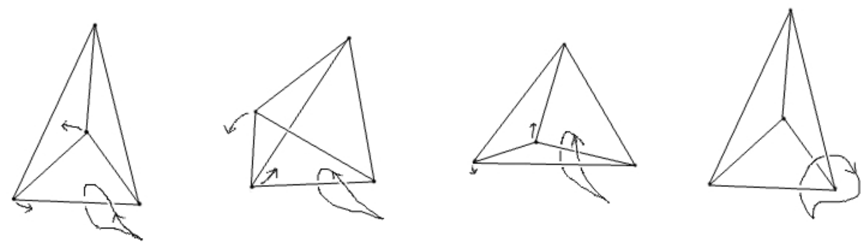

Figure 1: Showing a generator which squares to a conjugation in $\mathrm{Aut}(F_3)$.

it was was shown to deformation retract to the case of linearly embedded unlinked circles. Taking this as a point of departure, my research involves computing homotopy groups of other configuration spaces $C(X, Y)$, through geometric and topological arguments, and in studying the action of its fundamental group on $\pi_1(Y \setminus X)$. The cases I've looked at all have free groups for $\pi_1(Y \setminus X)$, and so give families of subgroups of $\mathrm{Aut}(F_n)$.

A natural next case following the work of Brendle and Hatcher is to generalize from the configuration space of unlinked circles in $\mathbb{R}^3$ to (a particular path component of) the configuration space of a graph in $\mathbb{R}^3$. The cases of linear configurations of complete graphs $K_4$ and $K_5$ in $\mathbb{R}^3$ are included in the more generalized result of my first paper. For $K$ the $(n-2)$-skeleton of the $n$-simplex, and $L$ the $(n-2)$-skeleton of the $(n+1)$-simplex, I have shown that their configuration spaces in $\mathbb{R}^n$, denoted $C(K)$ and $C(L)$, are homotopy equivalent to the double mapping cylinder

$$SO(n)/A_{n+1} \leftarrow SO(n)/A_n \rightarrow SO(n)/S_n,$$

where $S_n$, $A_n$ are the symmetric and alternating groups. This first paper of mine, arxiv:1403.1850v2, submitted for publication in Geometriae Dedicata in Jan 2015, gives a few methods of computing this, as well gives the homotopy type of the covering space of embeddings, and for the case $n = 3$ presents the fundamental group and its action on $F_3$, the fundamental group of the complement of an unknotted tetrahedral graph in $\mathbb{R}^3$ (See figure 1).

The essence of the homotopy equivalence between the spaces $C(K)$ and $C(L)$ comes down to the 3 types of symmetries found in Figure 2, which have similar analogs in higher dimensions.

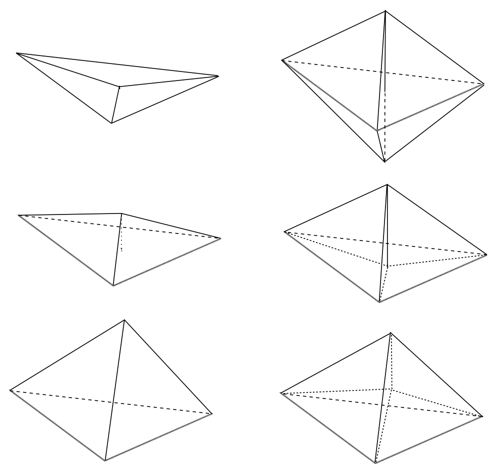

Figure 2: On the left, $K_4$ pyramids, from top down: degenerate, half-high, regular; on the right, $K_5$, from top down: dihedral, point interior to face, point interior to $K_4$. Note the equality of the symmetries between horizontally adjacent figures.

Currently, I am nearing the end of a second paper on the $C^\infty$ embeddings of planar graphs (those in the unknotted component) (Here, $C^\infty$ means that each closed edge is $C^\infty$ embedded). The primary tool used here is Hatcher's affirmative proof of the Smale conjecture which states, among many equivalences, that $\mathrm{Diff}(S^3)$ deformation retracts to $O(4)$. The theorem is used to rigidify the embeddings, so the space of $C^\infty$ embeddings has a finite dimensional model. I expect a full description of the homotopy type of this embedding space to be available Febuary 2015 and will include methods on understanding configuration spaces of graphs with symmetries.

There are other similar problems I am interested in pursuing. It is possible, for example, that the 2 higher dimensional complexes cases previously considered could be extended to handle $k$ disjoint unlinked complexes without great difficulty. It would also be interesting to approach the linear case of $k$-skeleta of $m$-simplices in $\mathbb{R}^n$ for other values $k, m, n$.

My hobby interests include category theory, expander graphs, experimental topology (writing software to investigate topological claims) and philosophy of mathematics, in addition to my broad interests across algebraic topology. As I produce results on problems similar to my progress, thus far, I hope to develop understanding in other peripheral areas and expand what I am able to contribute to the literature and to the mathematical community.