Lec 10: Elementary matrices and their products

Any elementary row transformation of a matrix A can be represented by a square matrix E such that applying this transformation to A is the same as multiplying A by E from the left: EA. Consider some examples. Take 3×4 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and interchange its first and second rows. We get

$$B = A_{r_1 \leftrightarrow r_2} = \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}.$$

Now notice that $B = EA$ for some matrix E:

$$\begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}.$$

We see that interchanging of the first and second rows can be represented by matrix $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Similarly, interchanging of top and bottom rows is represented by matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix}.$$

Relation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ ra_{21} & ra_{22} & ra_{23} & ra_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

shows that the scalar multiplication by r of the second row is represented by matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{bmatrix}$. What about the third type elementary row transformations? Say, adding c times top row to the second one? Here we go:

$$\begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ ca_{11} + a_{21} & ca_{12} + a_{22} & ca_{13} + a_{23} & ca_{14} + a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}.$$

So, this transformation is represented by matrix $\begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Matrix $\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ corresponds to adding c times third row to the first one. Indeed,

$$\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} ca_{31} + a_{11} & ca_{32} + a_{12} & ca_{33} + a_{13} & ca_{34} + a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}.$$

All these matrices corresponding to elementary row transformations are called *elementary matrices*. Similarly they can be described for any $m \times n$ matrix A. Note that elementary matrices are all square of order m. Moreover, they are all invertible. [This is because elementary transformations are invertible.] For elementary matrices above we have (verify!):

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If a matrix B is obtained from A by an elementary row transformation (ERT), then, as we already know, $B = E_1 A$ for some elementary matrix E_1. Now if C is obtained from B by an ERT, then $C = E_2 B$ for some E_2. Therefore $C = E_2 B = E_2(E_1 A) = (E_2 E_1)A$. We see that a composition of ERTs is represented by the product of corresponding elementary matrices (in the reverse order). In general, if B is obtained from A be a sequence of ERT with matrices $E_1, E_2, \dots, E_k$, then $B = (E_k E_{k-1} \cdots E_1)A$.

Now, we know that any matrix A can be transformed by a sequence of ERT to a matrix B in a reduced row echelon form (RREF). Then $B = EA$ where E is a product of elementary matrices. In particular, E is invertible. If A is invertible, then so is B as a product of two invertible matrices. But a matrix in a RREF is invertible if and only if it is the identity matrix I_n (why?). So it must be $EA = B = I_n$. Then E is an inverse of A. Thus we have proved

Theorem. *An $n \times n$ matrix is invertible if and only if its reduced row echelon form is identity matrix I_n.*