Lec 11: Finding the inverse

We know that an elementary row transformation (ERT) can be represented by an elementary matrix E (that is, the applying of this ERT to a matrix A gives the product EA). An easy way to remember which E corresponds to which ERT is simply to apply the ERT to the identity matrix I_n: what we obtain is $EI_n = E$, i. e. exactly the corresponding elementary matrix E. For instance, in case $n = 3$, interchanging first and second rows of I_3 gives $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and from the last lecture we know that this matrix represents the ERT (interchange of first and second rows). Multiplication by 2 of the last row of I_3 and subtracting 3 times the last row of I_3 from the first one produce respectively matrices

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

More generally, the matrix E representing the sequence of elementary row transformations with matrices $E_1, E_2, \dots E_k$, can either be found as the product $E_k E_{k-1} \cdots E_1$ or as application of this sequence to I_n. Indeed, we have $E_k(E_{k-1}(\cdots E_2(E_1 I_n))) = (E_k E_{k-1} \cdots E_1) I_n = E_k E_{k-1} \cdots E_1$.

Now let A be an $n \times n$ matrix. Suppose it is invertible, hence its reduced row echelon form (RREF) is I_n and $A^{-1} = E_k E_{k-1} \cdots E_1$. Here, $\{E_i\}_{i=1}^k$ is a sequence of ERT producing RREF from A.[1] This observation suggests an algorithm of finding A^{-1}. We adjoin the matrix I_n to A and so consider the partitioned matrix $[A|I_n]$ (of size $n \times 2n$). Now apply to this matrix the sequence of ERT producing RREF from A. Thus in the left part of the partitioned matrix we'll get I_n and in the right one $E_k E_{k-1} \cdots E_1$, i. e. A^{-1}. In other words, we'll get $[I_n|A^{-1}]$.

If we don't know whether A is invertible, we still can apply the same algorithm to $[A|I_n]$. Let's transform this matrix by ERTs until we obtain the RREF of A on the left side. If it is I_n, great, A is invertible and on the right side it must be A^{-1}. If the RREF differs from I_n, then we conclude that A is singular (noninvertible). The right side of the partitioned matrix is then not so valuable.

Example. Determine whether a matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

[1] In fact, before we only showed that $EA = I_n$ where $E = E_k E_{k-1} \cdots E_1$. But for E to be the inverse to A we need also $AE = I_n$. To prove the latter, multiply the relation $EA = I_n$ by E from the right: $EAE = E$. Now, E is invertible, so let's multiply the latter relation by E^{-1} from the left: $E^{-1}EAE = E^{-1}E$, or $AE = I_n$, what was required. Hence $A^{-1} = E$.

is invertible or not. If yes, find A^{-1}.

Consider the partitioned matrix $[A|I_3]$:

$$B = \left[\begin{array}{rrr|rrr} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right].$$

Apply ERTs to B which transform A to RREF:

$$C = B_{2r_1 \to r_1} = \left[\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right], \quad D = C_{r_1 + r_2 \to r_2} = \left[\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right],$$

$$E = D_{\frac{3}{2}r_2 \to r_2} = \left[\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right], \quad F = E_{r_2 + r_3 \to r_3} = \left[\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right],$$

$$G = F_{\frac{4}{3}r_3 \to r_3} = \left[\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right], \quad H = G_{\frac{2}{3}r_3 + r_2 \to r_2} = \left[\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right],$$

$$I = H_{\frac{1}{2}r_2 + r_1 \to r_1} = \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right].$$

We see that RREF of A is I_3, so A is invertible and

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}.$$

Note that an $n \times n$ matrix B having a row of zeros is singular. Hence if B is obtained from A by a sequence of ERT, then A must be singular. Indeed, $B = EA$ (where E is a product of elementary matrices) and invertibility of A would imply that of B (as the product of invertible matrices). But B is singular, therefore A is singular. In particular, if in the process of finding A^{-1} from $[A|I_n]$ we get $[B|C]$ where B has a zero row, we can stop and say that A is singular. [Then there is no need to continue transforming A to RREF.]

Example. Determine whether the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

is invertible or not. If yes, find A^{-1}.

Again we work with the partitioned matrix $B = [A|I_3]$ and apply to B the sequence of ERT transforming A to it's RREF.

$$C = B_{-2r_1+r_2\to r_2} = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -5 & -2 & 1 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 \end{array}\right],$$

$$D = C_{r_2+r_3\to r_3} = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -5 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right].$$

Since the left submatrix of D has a zero row, A is noninvertible.

In previous classes we proved small theorems which can be included in a bigger one.

Theorem. *Let A be an $n \times n$ matrix. The following statements are equivalent:*

(1) A is invertible.

(2) The linear system $A\bar{x} = \bar{b}$ has a unique solution for every $\bar{b}$.

(3) The homogeneous system $A\bar{x} = 0$ has only the trivial solution.

(4) The RREF of A is I_n.

(5) A is a product of elementary matrices.

We prove this theorem again in order to demonstrate how one can show equivalence of many statements without going through all pairs of them. We will prove implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1) and thus the equivalence of all statements.

Consider (1) $\Rightarrow$ (2) first. If A is invertible, then we can multiply by A^{-1} the system $A\bar{x} = \bar{b}$ and get $\bar{x} = A^{-1}\bar{b}$. So, if the solution exists, it is unique and is equal to $A^{-1}\bar{b}$. On the other hand, $\bar{x} = A^{-1}\bar{b}$ is a solution, because satisfies to the equation. Thus we've proved (1) $\Rightarrow$ (2). Now (2) implies (3) if we take $\bar{b}$ consisting of all zeros.

Prove (3) $\Rightarrow$ (4). According to the Gauss-Jordan reduction, the system $A\bar{x} = 0$ is equivalent to $A'\bar{x} = 0$, where A' is the RREF of A. But if $A' \neq I_n$, A' has a zero row at bottom (why?), which corresponds to the equation $0x_n = 0$. Then x_n can be any and the system $A'\bar{x} = 0$ has infinitely many solutions. Therefore so does $A\bar{x} = 0$. This contradiction shows that $A' = I_n$.

If RREF of A is I_n, then $(E_k E_{k-1} \cdots E_1)A = I_n$ for some elementary matrices E_i. Multiplying by $E_1^{-1}E_2^{-1} \cdots E_k^{-1}$ we obtain $A = E_1^{-1}E_2^{-1} \cdots E_k^{-1}$. Since inverses to elementary matrices are elementary matrices, A is a product of elementary matrices, and we have the implication (4) $\Rightarrow$ (5) proved. Finally, implication (5) $\Rightarrow$ (1) follows from the fact that a product of invertible matrices is invertible.