Lec 12: Elementary column transformations and equivalent matrices

Like ERT, one can define *elementary column transformations (ECT)*. These are:

- Interchanging two columns.
- Multiplying a column by a nonzero scalar.
- Adding a multiple of a column to another column.

Using familiar arguments, one can show that any matrix can be transformed to a *column echelon form (CEF)* by sequence of ECT. We say that a matrix is in CEF, if

- All zero columns are on the right.
- The first (if we go down from the top) nonzero entry in each column is 1 (the *leading one* of the column).
- If $j > i$, then the leading one of the column c_j appears below that of c_i.

The following matrices are in CEF:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

and these are <u>not</u> in CEF (why?):

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

A matrix is in a *reduced column echelon form (RCEF)* if it is in CEF and, additionally, any row containing the leading one of a column consists of all zeros except this leading one. In examples of matrices in CEF above, first and third matrices are in RCEF, and the second is not. Like row case, one can produce (a unique) RCEF for any matrix.

<u>Important note</u>. Applying column transformations is not allowed in solving linear systems.[1]

To ECTs there correspond elementary matrices, but unlike row case, they are multiplied from the right. That is, if B is obtained from A by an ECT, then $B = AF$ (not FA, as for ERT!). Matrix F is square of order n, where n is the number of columns of A. All three types of ECT are represented by their matrices below (interchanging first two columns, multiplying third column by r, adding c times second column to the first one):

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \end{bmatrix},$$

[1] Despite this note, ECTs may be useful sometimes as we'll see later.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & ra_{13} \\ a_{21} & a_{22} & ra_{23} \end{bmatrix},$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} + ca_{12} & a_{12} & a_{13} \\ a_{21} + ca_{22} & a_{22} & a_{23} \end{bmatrix}.$$

The elementary matrix corresponding to ECT is the matrix obtained from the identity matrix by this ECT. If B is obtained from A by an ECT and C is obtained from B by an ECT, then we have $B = AF_1$, $C = BF_2 = (AF_1)F_2 = A(F_1F_2)$. Hence, to the sequence of ECTs with matrices $F_1, F_2, \ldots, F_k$ it corresponds the matrix $F = F_1F_2\cdots F_k$ (multiplying in straight order, unlike the row case). We can find F either by straight multiplication of F_i or by applying the sequence of ECTs to the identity matrix I_n.

Now return awhile to row transformations. Matrix B is said to be *row equivalent* to matrix A, if B is produced from A by a sequence of ERTs. For example, A is row equivalent to itself (empty sequence of ERTs). Statement "B is row equivalent to A" means $B = (E_k \cdots E_2E_1)A$ for some elementary matrices E_i. Or, what is the same, $A = (E_1^{-1}E_2^{-1} \cdots E_k^{-1})B$. Since inverses of elementary matrices are elementary again, A is row equivalent to B. Thus, the relation of being row equivalent is symmetric, and instead of "B is row equivalent to A" we can say "A and B are row equivalent". [This reflects the fact that if B is produced from A by a sequence of ERTs, then A is produced from B by a sequence of ERTs.] As we know, any matrix is row equivalent to a matrix in REF, or a square matrix is invertible if and only of it is row equivalent to the identity matrix. If A and B are row equivalent, and B and C are row equivalent, then A and C are row equivalent (why?). The latter property is called *transitivity.*

By analogy, B is *column equivalent* to A, if B is produced from A by a sequence of ECTs. Or, what is the same, $B = A(F_1F_2\cdots F_k)$. Like above, if B is column equivalent to A, then A is column equivalent to B. Hence we can say that A and B are column equivalent. Any matrix is column equivalent to a matrix in CEF and a square matrix is invertible if and only of it is column equivalent to the identity matrix.

Now, a more general definition. Matrix B is *equivalent* to A, if B is obtained from A by a sequence of ERT and ECT. This means that for some elementary matrices $E_1, E_2, \ldots, E_k$, $F_1, F_2, \ldots, F_l$ we have $B = (E_k \cdots E_2E_1)A(F_1F_2\cdots F_l)$. Then A is equivalent to B, because after multiplying the latter relation by $(E_1^{-1} \cdots E_k^{-1})$ on the left and by $(F_l^{-1} \cdots F_1^{-1})$ on the right, we get $A = (E_1^{-1} \cdots E_k^{-1})B(F_l^{-1} \cdots F_1^{-1})$. Any matrix is equivalent to itself, and if two matrices are row (or column) equivalent, then they are equivalent. The relation of being equivalent is transitive.

Theorem. *Any $m \times n$ matrix A is equivalent to a (unique) partitioned matrix of the form*[2]

$$\begin{bmatrix} I_r & O_{r\ n-r} \\ O_{m-r\ r} & O_{m-r\ n-r}. \end{bmatrix} \tag{1}$$

[2] O_{kl} stands for the zero $k \times l$ matrix.

We will not prove this theorem. [Try to do it yourself[3] or see the proof in the book, p. 127. Uniqueness of the form (1) is not proved though.] Instead let's look at the following example.

Example. Transform the matrix

$$A = \begin{bmatrix} 2 & 4 & 7 \\ 1 & 2 & 3 \end{bmatrix}$$

to a matrix of form (1) using ERTs and ECTs.

First, let's produce the RREF of A by ERTs:

$$B = A_{r_1 \leftrightarrow r_2} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \end{bmatrix}, \quad C = B_{r_2 - 2r_1 \to r_2} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = C_{r_1 - 3r_2 \to r_1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix D is in RREF but still not in form (1). Interchange its last two columns and then subtract two times the first column from the last one:

$$G = D_{c_2 \leftrightarrow c_3} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}, \quad H = G_{c_3 - 2c_1 \to c_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Now matrix H is of the required form.

By this example we've shown that $H = EAF$ where E and F are products of elementary matrices corresponding to the ERTs and ECTs we've performed. Let's find E and F. Matrix E is the matrix obtained from I_2 (I_2 because A has two rows) by the sequence of ERTs we used. Namely, $r_1 \leftrightarrow r_2, r_2 - 2r_1 \to r_2$ and $r_1 - 3r_2 \to r_1$. Applying this sequence to the identity I_2, we obtain (verify!)

$$E = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}.$$

Similarly, applying the sequence of ECTs we've used (i. e. $c_2 \leftrightarrow c_3$ and $c_3 - 2c_1 \to c_3$), to matrix I_3 (3 because A has three columns), we compute (verify!)

$$F = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

If you like to multiply matrices, find the product EAF and make sure it coincides with H.

Note that invertible matrix is equivalent to the identity (it is even row equivalent). Conversely, if a matrix A is equivalent to I_n, it must be invertible. Indeed, $A = EI_nF = EF$ and E, F are invertible as products of elementary matrices. Thus we have a nice way to check whether a matrix A is invertible: transform it by ERTs and ECTs to a form (1) and see if it is the identity (if it's not I_n, then A is singular). Note that transforming A to form (1) reveals only the fact of invertibility of A but, generally, this way can't be used to find A^{-1}.

[3]Hint: transform a matrix to RREF using ERTs and then get the form (1) by ECTs.