Lec 14: Determinants of matrices

Determinants are defined for square matrices only. It is just a number associated with a matrix A. This number is denoted by $\det(A)$. By definition, if $A = [a]$ is a 1×1 matrix, $\det(A)$ is the entry of A, i. e. $\det([a]) = a$. To define $\det(A)$ when $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is of order 2, we consider one example which will serve as motivation.

Let's find the area of a parallelogram $OABC$ generated by vectors $\bar{OA} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\bar{OC} = \begin{bmatrix} c \\ d \end{bmatrix}$ as shown on Figure 1. [Coordinates of points A and C are respectively (a, b) and (c, d).]

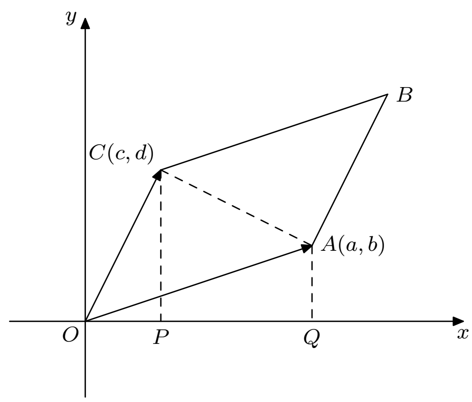

Figure 1: Parallelogram generated by vectors $\bar{OA}$ and $\bar{OC}$

The area of $OABC$ is twice the area of the triangle OAC, and

$$\text{Area}(OAC) = \text{Area}(OPC) + \text{Area}(PQAC) - \text{Area}(OQA) \tag{1}$$

We shall use the formulas for areas of triangles and trapezoids. We have

$$\text{Area}(OPC) = \frac{|OP||PC|}{2} = \frac{cd}{2}, \quad \text{Area}(OQA) = \frac{ab}{2},$$

$$\text{Area}(PQAC) = |PQ|\frac{|CP| + |AQ|}{2} = (a-c)\frac{d+b}{2}.$$

Plugging these to formula (1), we obtain after simple manipulations: $\text{Area}(OAC) = \frac{ad-bc}{2}$, hence

$$\text{Area}(OABC) = ad - bc. \tag{0.1}$$

Note that the number $ad - bc$ may be negative. It is positive for our picture, but if $\bar{OA}$ were above $\bar{OC}$, then it would be negative. So to be correct we should have written $|ad - bc|$ because areas are positive. Without taking absolute value, we have not area but *oriented area* which may be negative.

For a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

we define

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}. \tag{2}$$

By the example above, $\det(A)$ is the (oriented) area of a parallelogram generated by columns of A.

A 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

has the determinant

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}. \tag{3}$$

Similarly, one can show that this is the area of a parallelepiped generated by columns of A. [Don't try to do this!] Remember formulas (2) and (3). A simple way to remember formula (3) will be shown in class.

For example,

$$\det([3]) = 3, \quad \det(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}) = 1 \cdot 4 - 3 \cdot 2 = -2,$$

$$\det(\begin{bmatrix} 0 & -1 & 2 \\ 1 & 3 & 4 \\ -2 & 5 & 0 \end{bmatrix}) = 0{\cdot}3{\cdot}0 + (-1){\cdot}4{\cdot}(-2) + 1{\cdot}5{\cdot}2 - 2{\cdot}3{\cdot}(-2) - 0{\cdot}4{\cdot}5 - (-1){\cdot}1{\cdot}0 = 30.$$

Hence the area of a parallelogram generated by vectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is $|-2| = 2$ square units. The area of a parallelepiped generated by columns of the last matrix is $|30| = 30$ cubic units.

Before giving a general definition of the determinant, let's talk about permutations. A *permutation* on the set $S = \{1, 2, \ldots, n\}$ is a rearrangement $j_1 j_2 \ldots j_n$ of its elements. Here j_1 is a number where 1 goes, j_2 is the image of 2, etc. For example, 2431 is a permutation on $S = \{1, 2, 3, 4\}$ which sends $1, 2, 3, 4$ to $2, 4, 3, 1$ respectively ($j_1 = 2, j_2 = 4, j_3 = 3, j_4 = 1$). In particular, it does not move element 3. Permutation $12 \ldots n$ (i. e. $j_i = i$) preserves all elements on their places. It is called *trivial.* Of course, all numbers in the permutation $j_1 j_2 \ldots j_n$ are different since they are different in S. How many permutations are there for the set S? Try to create a permutation. We can move 1 to any number, so there are n options for j_1. After that, j_2 can be any different from j_1, hence there are $n-1$ options for j_2. Number j_3 can be any but j_1 and j_2 because those places are already occupied. We conclude that there are $n-2$ possibilities to choose j_3. At this stage, we have $n(n-1)(n-2)$ options for the starting 3 numbers $j_1 j_2 j_3$ of our permutation. Finally, there are $n(n-1)(n-2)(n-3)\cdots 2\cdot 1 = n!$ variants for the whole permutation $j_1 j_2 \dots j_n$. We conclude that there are exactly $n!$ permutations on the set of n elements. The set of all permutations on S is denoted by S_n.

If a larger number j_r precedes a smaller one j_s, then we say that elements (j_r, j_s) are an *inversion*. A permutation $j_1 j_2 \dots j_n$ is called *even*, if it has even number of inversions, and *odd*, if the number of its inversions is odd. One can show that the amounts of even and odd inversions are the same, i. e. equal to $\frac{n!}{2}$ (if $n > 1$). The permutation 2431 has 4 inversions: $(2,1)$, $(4,3)$, $(4,1)$ and $(3,1)$, hence it is even. The permutation 231 is odd, because it has 3 inversions: $(2,3)$, $(2,1)$ and $(3,1)$ (in this case all pairs of elements are inversions). The trivial permutation is even since it contains 0 inversions. The permutation $21345\dots n$ (switching 1 and 2) is odd for it only contains one inversion $(2,1)$. In general, if a permutation transposes only two numbers and leaves the rest on their places, it is odd. [Try to explain it.]

Now let $A = [a_{ij}]$ be an $n \times n$ matrix. The determinant of A is

$$\det(A) = \sum (\pm) a_{1j_1} a_{2j_2} \cdots a_{nj_n} \tag{4}$$

where the summation is over all permutations $j_1 j_2 \dots j_n$. We take the sign $+$ before the term $a_{1j_1} a_{2j_2} \cdots a_{nj_n}$ if the permutation $j_1 j_2 \dots j_n$ is even, and $-$ if the permutation is odd. In case $n = 1$ we have $\det(A) = a_{11}$. For $n = 2$ we obtain formula (2) as S_2 contains only two permutations: 12 and 21, the latter being odd. A little bit harder to verify (do it!) that for $n = 3$ formula (4) is the same as (3). You may regard det as a volume of a parallelepiped in space of many dimensions.

Formula (4) contains $n!$ terms in the sum since it is over all permutations. So for larger n, i. e. $n = 4, 5, 6, \dots$ we have $24, 120, 720, \dots$ terms to sum. This is extremely boring. On the next lecture we will discover some properties of determinants which will allow to compute them faster.