Lec 16: Cofactor expansion and other properties of determinants

We already know two methods for computing determinants. The first one is simply by definition. It works great for matrices of order 2 and 3. Another method is producing an upper-triangular or lower-triangular form of a matrix by a sequence of elementary row and column transformations. This can be performed without much difficulty for matrices of order 3 and 4. For matrices of order 4 and higher, perhaps, the most efficient way to calculate determinants is the *cofactor expansion*. This method is described as follows.

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Denote by M_{ij} the submatrix of A obtained by deleting its row and column containing a_{ij} (that is, row i and column j). Then $\det(M_{ij})$ is called the *minor* of a_{ij}. For example, let

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}. \tag{1}$$

M_{11} is obtained by deleting row 1 and column 1; M_{23} is A without row 2 and column 3:

$$M_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} \quad M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}.$$

The minor of a_{11} is $\det(M_{11}) = 5\cdot 9 - 8\cdot 6 = -3$ and the minor of a_{23} is $\det(M_{23}) = -6$.

If we multiply the minor of a_{ij} by $(-1)^{i+j}$, then we arrive at the definition of the *cofactor* A_{ij} of a_{ij}:

$$A_{ij} = (-1)^{i+j} \det(M_{ij}).$$

In the example above, $A_{11} = (-1)^2 \cdot (-3) = -3$, $A_{23} = (-1)^5 \cdot (-6) = 6$. Verify that $A_{12} = 6$, $A_{13} = -3$ and find the rest of cofactors.

The method of cofactor expansion is given by the formulas

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \quad \text{(expansion of } \det(A) \text{ along } i^{\text{th}} \text{ row)}$$

$$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} \quad \text{(expansion of } \det(A) \text{ along } j^{\text{th}} \text{ column)}$$

Let's find $\det(A)$ for matrix (1) using expansion along the top row:

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 1 \cdot (-3) + 2 \cdot 6 + 3 \cdot (-3) = 0.$$

[Compare with the first example from the previous lecture. Basing on that example, could you say that $\det(A) = 0$ without any calculations?] It would be the same as if we used the expansion along any other row or column. For example, the expansion along the second column gives:

$$\det(A) = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} = -2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 8\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 0.$$

The method of cofactor expansion is especially applicable if a matrix has a row or a column with many zeros. Then we expand the determinant along this row or column.

Example. Compute the determinant of

$$A = \begin{bmatrix} 2 & -1 & 1 & 0 \\ 3 & 5 & 0 & -2 \\ 1 & 1 & 0 & -3 \\ 4 & 0 & 3 & -1 \end{bmatrix}.$$

The third column looks more preferable as it contains two zeros. Let's use the expansion along this column.

$$\begin{vmatrix} 2 & -1 & 1 & 0 \\ 3 & 5 & 0 & -2 \\ 1 & 1 & 0 & -3 \\ 4 & 0 & 3 & -1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & 5 & -2 \\ 1 & 1 & -3 \\ 4 & 0 & -1 \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & -1 & 0 \\ 3 & 5 & -2 \\ 1 & 1 & -3 \end{vmatrix} = -50 + 99 = 49.$$

[We omitted zero terms.] Note that for computing the 3×3 determinants above we can use the expansion again. For example

$$\begin{vmatrix} 3 & 5 & -2 \\ 1 & 1 & -3 \\ 4 & 0 & -1 \end{vmatrix} = 4 \begin{vmatrix} 5 & -2 \\ 1 & -3 \end{vmatrix} + (-1) \begin{vmatrix} 3 & 5 \\ 1 & 1 \end{vmatrix} = -52 + 2 = -50.$$

[We used the expansion along the bottom row.]
Important exercise: find $\det(A)$ expanding along the second row and make sure the answer is the same.

Now let's discuss some questions regarding determinants.

- What are the determinants of elementary matrices? They are -1, r and 1 for elementary matrices of respectively first, second (multiplication of a row or a column by r) and third type. For example, let E be an elementary matrix corresponding to switching two rows. If we apply this ERT to the identity matrix I_n, we get $EI_n = E$. On the other hand, from the previous lecture we know that det is multiplied by -1 after this transformation: $\det(E) = -\det(I_n) = -1$.
- Is it true that $\det(rA) = r\det(A)$? Yes — see the previous lecture.
- Is it true that $\det(AB) = \det(A)\det(B)$? Yes. See the proof on pp. 151—153 of the book. As a consequence, the determinant of a product of any number of matrices is equal to the product of their determinants.
- Is it true that $\det(A+B) = \det(A) + \det(B)$? No. If we take $A = I_2$, $B = -I_2$, then $\det(A+B) = 0$ but $\det(A) = \det(B) = 1$, and $0 \neq 1 + 1 = 2$.

The property $\det(AB) = \det(A)\det(B)$ is very important. It allows to prove

Theorem. *Matrix A is invertible if and only if* $\det(A) \neq 0$.

Proof. If A is invertible, then it is a product of elementary matrices. Then, by the mentioned property, the determinant of A is product of determinants of these matrices. Each of these determinants is nonzero as it must be -1, $r \neq 0$ or 1. Therefore $\det(A) \neq 0$. On the other hand, if A is singular, its RREF $B = EA$ has a row of zeros, and $0 = \det(B) = \det(E)\det(A)$. Since $\det(E) \neq 0$, $\det(A) = 0$. □

If A^{-1} exists, then $AA^{-1} = I_n$ and $\det(A)\det(A^{-1}) = \det(I_n) = 1$. Hence $\det(A^{-1}) = \frac{1}{\det(A)}$. By theorem, matrix (1) is singular and the 4×4 matrix in the example above is invertible, with the determinant $\frac{1}{49}$.