## Lec 19, 20: Vector spaces

The name "vector space" conjures up, perhaps, the image of directed line segments from the plane or 3 -space. Another way to see it is the set of $m \times 1$ matrices for some fixed $m$ (our first definition of vectors), and even another way is the set of sequences $\left(x_{1}, x_{2}, \ldots, x_{m}\right)$ (coordinates of points corresponding to vectors). If we prove a theorem about the set of $m \times 1$ matrices, then we need to prove the same theorem for the sequences if we want to treat vectors as the sequences. In order to avoid multiple theorems which are essentially the same, we instead prove a theorem once for vector spaces, which encompass both matrices and sequences (and a lot more).

Roughly speaking, a vector space is a set, elements of which one can add and multiply by a scalar, with usual properties of addition and multiplication satisfied. For example, the set of all $m \times n$ matrices and the set of all polynomials are vector spaces. The idea is to unify objects having many properties in common. To be more precise,
a vector space is a set $V$ of elements ("vectors") on which we have two operations $\oplus$ ("addition") and $\odot($ "multiplication by a scalar") defined with the following properties:

- if $\mathbf{u}$ and $\mathbf{v}$ are in $V$, then $\mathbf{u} \oplus \mathbf{v}$ is in $V$ and
(1) $\mathbf{u} \oplus \mathbf{v}=\mathbf{v} \oplus \mathbf{u}$ for all $\mathbf{u}, \mathbf{v}$ in $V$,
(2) $\mathbf{u} \oplus(\mathbf{v} \oplus \mathbf{w})=(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $V$,
(3) there exists an element $\mathbf{0}$ ("zero vector") in $V$ such that $\mathbf{u} \oplus \mathbf{0}=\mathbf{0} \oplus \mathbf{u}=\mathbf{u}$ for all $\mathbf{u}$ in $V$,
(4) for each $\mathbf{u}$ in $V$ there exists an element $-\mathbf{u}$ ("negative of $\mathbf{u}$ ") in $V$ such that $\mathbf{u} \oplus-\mathbf{u}=-\mathbf{u} \oplus \mathbf{u}=\mathbf{0}$;
- if $\mathbf{u}$ is in $V$ and $c$ is a real number, then $c \odot \mathbf{u}$ is in $V$ and
(5) $c \odot(\mathbf{u} \oplus \mathbf{v})=c \odot \mathbf{u} \oplus c \odot \mathbf{v}$ for all $\mathbf{u}, \mathbf{v}$ in $V$ and $c$ in $\mathbb{R}$,
(6) $(c+d) \odot \mathbf{u}=c \odot \mathbf{u} \oplus d \odot \mathbf{u}$ for all $\mathbf{u}$ in $V$ and $c, d$ in $\mathbb{R}$,
(7) $c \odot(d \odot \mathbf{u})=(c d) \odot \mathbf{u}$ for all $\mathbf{u}$ in $V$ and $c, d$ in $\mathbb{R}$,
(8) $1 \odot \mathbf{u}=\mathbf{u}$ for all $\mathbf{u}$ in $V$.

Examples of vector spaces (verify properties (1)-(8)!):

- $V$ is the set of arrows on the plane issued from the same point (say, $(0,0)$ ). The sum $\mathbf{u} \oplus \mathbf{v}$ of arrows $\mathbf{u}$ and $\mathbf{v}$ is the arrow with head at a vertice of the parallelogram constructed by $\mathbf{u}$ and $\mathbf{v}$ as shown on Figure 1. The product $c \odot \mathbf{u}$ is the scaling of vector $\mathbf{u}$ by $c$.
- $V$ is the set of sequences $\mathbf{u}=\left(u_{1}, u_{2}, \ldots, u_{n}\right)$. We define $\left(u_{1}, u_{2}, \ldots, u_{n}\right) \oplus$ $\left(v_{1}, v_{2}, \ldots, v_{n}\right)=\left(u_{1}+v_{1}, u_{2}+v_{2}, \ldots, u_{n}+v_{n}\right)$ and $c \odot\left(u_{1}, u_{2}, \ldots, u_{n}\right)=$ $\left(c u_{1}, c u_{2}, \ldots, c u_{n}\right)$. Then $V$ is a vector space. In particular, $\mathbf{0}=(0,0, \ldots, 0)$ and $-\mathbf{u}=\left(-u_{1},-u_{2}, \ldots,-u_{n}\right)$. If $n=1$, then $V$ is just $\mathbb{R}$ with usual operations.

Figure 1: Operations with vectors

- $V$ is the set of all $m \times n$ matrices with usual addition and multiplication by a scalar: $A \oplus B=A+B$ and $c \odot A=c A .{ }^{1}$
- $V$ is the set of all polynomials with usual addition and multiplication by a scalar: $f(t) \oplus g(t)=f(t)+g(t)$ and $c \odot f(t)=c f(t)$.
- $V$ is the set of all continuous functions on the interval $[0,1]$ with usual addition and multiplication by a scalar.

Examples of not vector spaces:

- $V$ is the set of all $m \times n$ matrices with operations $A \oplus B=A-B, c \odot A=c A$. [Our "addition" is the usual subtraction.] Then $V$ is not a vector space because the property (1) does not hold. Indeed, it must be $A \oplus B=B \oplus A$ which is $A-B=B-A$. But the latter means $A=B$, and if we take different $A$ and $B$, this condition fails.
- $V$ is the set of real numbers with operations $u \oplus v=u v, c \odot u=c u$. ["Addition" is the usual multiplication.] On one hand $0 \odot u=0 u=0$ and, on the other hand, it must be $0 \odot u=(1+(-1)) \odot u=1 \odot u \oplus(-1) \odot u$ (property (6)) $=u(-u)=-u^{2}$. Hence $0=-u^{2}$ which fails once we take a nonzero number $u$. Hence $V$ is not a vector space.
- $V$ is the set of real numbers with operations $u \oplus v=u+v, c \odot u=e^{c} u$. ["Multiplication by $c$ " is usual multiplication by the exponent $e^{c}$.] Then the property (8) is invalid, because $1 \odot u=e u$ while it must be $u$. Thus $V$ is not a vector space.

Remarkably, properties (1)-(8) imply many others, such as
(a) $0 \odot \mathbf{u}=\mathbf{0}$ for all $\mathbf{u}$ in V ,
(b) $c \odot \mathbf{0}=\mathbf{0}$ for all $c$ in $\mathbb{R}$,

[^0](c) $(-1) \odot \mathbf{u}=-\mathbf{u}$,
(d) $\mathbf{0}$ is unique,
(e) $-\mathbf{u}$ is unique for any $\mathbf{u}$ in $V$.

Prove (a). By property (6), we have

$$
0 \odot \mathbf{u}=(0+0) \odot \mathbf{u}=0 \odot \mathbf{u} \oplus 0 \odot \mathbf{u}
$$

By (4), there exists $-0 \odot \mathbf{u}$ such that $0 \odot \mathbf{u} \oplus-0 \odot \mathbf{u}=\mathbf{0}$. Then
$\mathbf{0}=0 \odot \mathbf{u} \oplus-0 \odot \mathbf{u}=(0 \odot \mathbf{u} \oplus 0 \odot \mathbf{u}) \oplus-0 \odot \mathbf{u}=0 \odot \mathbf{u} \oplus(0 \odot \mathbf{u} \oplus-0 \odot \mathbf{u})=0 \odot \mathbf{u} \oplus \mathbf{0}$
by property (2). By (3), $0 \odot \mathbf{u} \oplus \mathbf{0}=0 \odot \mathbf{u}$. Thus we have $\mathbf{0}=0 \odot \mathbf{u}$, which proves (a).

Let's prove (d). If there are two zero vectors $\mathbf{0}$ and $\mathbf{0}^{\prime}$, then

$$
\mathbf{0}=\mathbf{0} \oplus \mathbf{0}^{\prime}=0^{\prime}
$$

by property (3) applied first for $\mathbf{0}$ and then for $\mathbf{0}^{\prime}$. Thus $\mathbf{0}=\mathbf{0}^{\prime}$.
Exercise Prove (b), (c) and (e). Which properties in (1)-(8) do you use?
If a subset $W$ of a vector space $V$ is closed under the operations $\oplus$ and $\odot$, then we say that $W$ is a subspace of $V$. By "closed under the operations" we mean $\mathbf{u} \oplus \mathbf{v}$ and $c \odot \mathbf{u}$ belong to $W$ if $\mathbf{u}$ and $\mathbf{v}$ are in $W, c$ in $\mathbb{R}$. A trivial example of $W$ is the set consisting of the only element $\mathbf{0}$. This is called the zero subspace.

Every subspace $W$ must contain $-\mathbf{u}$ with every $\mathbf{u}$ in $W$. Indeed, $-\mathbf{u}=(-1) \odot \mathbf{u}$ belongs to $W$ as it is closed under multiplication by a scalar. Then $\mathbf{0}$ belongs to $W$, because so does $\mathbf{u}+(-\mathbf{u})$ which is $\mathbf{0}$. This shows, that any subspace is a vector space itself.

Examples:

- $V$ is the set of vectors in 3 -space with common origin $O$. The operations on $V$ are defined in a natural way, as on Figure 1. Let $W$ be a set of all vectors belonging to a plane $P$ which contains $O$. Then the sum $\oplus$ of vectors in $W$ belongs to $W$ again. The same with multiplication by a scalar. Hence $W$ is a subspace. A similar argument shows that if $P$ is a line through $O$, then $W$ is a subspace of $V$.
- Let $V$ be the set of $n \times 1$ matrices. If $A$ is an $m \times n$ matrix, then all solutions $\bar{x}$ of the homogeneous linear system

$$
A \bar{x}=0
$$

is a subset $W$ in $V$. Moreover, it is a subspace of $V$, since if $\bar{x}$ and $\bar{y}$ are solutions, then $\bar{x}+\bar{y}$ is a solution:

$$
A(\bar{x}+\bar{y})=A \bar{x}+A \bar{y}=0+0=0 .
$$

And $c \bar{x}$ is a solution as well:

$$
A(c \bar{x})=c(A \bar{x})=0 .
$$

Hence $W$ is a subspace. Note that if we took non-homogeneous system $A \bar{x}=\bar{b}$, then $W$ would not be a subspace since it doesn't contain $\mathbf{0}$.

- Let $V$ be the set of all $2 \times 2$ matrices $A=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]$. The trace $\operatorname{tr}$ of a matrix is the sum of all its diagonal elements. In our case $\operatorname{tr}(A)=a+d$. Let $W$ be the set of all traceless matrices in $V$ (i.e. of all $A$ with $\operatorname{tr}(A)=0$, or a=-d). Then $W$ is a subspace of $V$, because $\operatorname{tr}(A+B)=\operatorname{tr}(A)+\operatorname{tr}(B)=0$ and $\operatorname{tr}(c A)=c \operatorname{tr}(A)=0$ if $A, B$ in $W$ and $c$ in $\mathbb{R}$.
- $V$ is the vector space of all polynomials. If we add two polynomials of degree $\leq n$, then we obtain a polynomial of degree $\leq n$. The same with multiplication by a scalar. This shows that the set $W$ of all polynomials of degree less or equal to $n$ is a subspace of $V$.


[^0]:    ${ }^{1}$ As elements of $V$, these matrices are vectors in $V$. It may seem weird since we defined vectors as $m \times 1$ matrices. The explanation is that our first definition of vectors was irrelevant to any vector space. But in this case we stress that $m \times n$ matrices are vectors in $V$.

