Lec 21, 22: Span

Let V be a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ a set of its vectors. The set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a subspace in V (why?). This subspace is called the *span* of S or the span of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, and denoted by $\operatorname{Span} S$. Thus we have

$$\operatorname{Span} S = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k\} \quad \text{for all real numbers } a_1, \ldots, a_k.$$

[According to the previous lecture, we should have written $a_1 \odot \mathbf{v}_1 \oplus a_2 \odot \mathbf{v}_2 \oplus \cdots \oplus a_k \mathbf{v}_k$ for a linear combination. For the reason of notation simplicity, we will denote the operations $\oplus$ and $\odot$ by $+$ and $\cdot$ respectively. For instance, $a \odot \mathbf{u} \oplus b \odot \mathbf{v} = a\mathbf{u} + b\mathbf{v}$.]

For example, the span of matrices $\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$ and $\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}$ are all matrices of the form $\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$, or all diagonal matrices.

We define the span of the empty set as the trivial subspace $\{\mathbf{0}\}$.

Example. Let $V = \mathbb{R}^3$. Does vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$ belong to the span of $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$?

In other words, can $\mathbf{v}$ be represented as $a\mathbf{v}_1 + b\mathbf{v}_2$, or are there such numbers a and b that

$$a \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}?$$

This equation can be rewritten as a linear system with variables a, b:

$$2a + 3b = 1, a + b = 2, -2b = 6. \tag{1}$$

It has the solution $a = 5, b = -3$. Therefore the answer to our question is positive: $\mathbf{v}$ belongs to $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, specifically, $\mathbf{v} = 5\mathbf{v}_1 - 3\mathbf{v}_2$.

If the system (1) in our example had no solutions (say, if $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$), then the answer would be negative: $\mathbf{v}$ does not belong to $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Of course, the case of infinitely many solutions is also positive. The difference is that there are infinitely many linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ that equal $\mathbf{v}$ (not the only one as in the example). See also examples 7-10 of the book.

The span of 2-vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is the entire $\mathbb{R}^2$. For this we need to show that every $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ can be represented as $x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3$ for some x, y, z:

$$x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 2 \\ 3 \end{bmatrix} + z \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

We have the linear system with variables x, y, z:

$$x + 2y + 3z = a, 2x + 3y + 4z = b.$$

The RREF of the augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & |a \\ 2 & 3 & 4 & |b \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0 & -1 & |2b-3a \\ 0 & 1 & 2 & |2a-b. \end{bmatrix}$$

Then we have one free variable z, which can take any value, and $x = 2b - 3a + z$, $y = 2a - b - 2z$. So, any linear combination of the form $(2b - 3a + z)\mathbf{v}_1 + (2a - b - 2z)\mathbf{v}_2 + z\mathbf{v}_3$ equals $\mathbf{v}$. Hence $\mathbb{R}^2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. If we take $z = 0$, then $\mathbf{v} = (2b - 3a)\mathbf{v}_1 + (2a - b)\mathbf{v}_2$. This shows that $\mathbb{R}^2$ can be represented as the span of a less number of vectors:

$$\mathbb{R}^2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}.$$

Now we can't drop more vectors, that is $\text{Span}\{v_1\}$ and $\text{Span}\{v_2\}$ are less than $\mathbb{R}^2$ (why?). Such vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are called linearly independent. More precisely,
Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ and S_i is obtained from S by deleting $\mathbf{v}_i$. Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are called *linearly independent*, if $\text{Span}\, S_i \neq \text{Span}\, S$ (i. e. $\text{Span}\, S_i$ is contained in $\text{Span}\, S$ but does not coincide with it) for all $i = 1, \ldots, k$. So, in the last example $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent and vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly *dependent* (i. e. not linearly independent).

Theorem 0.1. *The following statements are equivalent:*

(1) $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are linearly independent,

(2) none $\boldsymbol{v}_i$ can be represented as a linear combination of other $\boldsymbol{v}_j$,

(3) if $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_k\boldsymbol{v}_k = \boldsymbol{0}$, then all a_i equal 0.

Proof. Prove the implication (1) $\Rightarrow$ (2) from the contrary. Suppose some $\mathbf{v}_i$ can be expressed through others, say,

$$\mathbf{v}_k = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_{k-1}\mathbf{v}_{k-1}.$$

Then $\text{Span}\, \mathbf{v}_1, \ldots, \mathbf{v}_k = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\}$, because any linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$ by plugging the formula above instead of $\mathbf{v}_k$. And this contradicts to linear independence of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

Now prove (2) $\Rightarrow$ (3), again from the contrary. Suppose $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$ and some $a_i \neq 0$. For certainty, let $a_k \neq 0$. Then we can divide the identity by a_k and express $\mathbf{v}_k = -\frac{a_1}{a_k}\mathbf{v}_1 - \frac{a_2}{a_k}\mathbf{v}_2 - \cdots - \frac{a_{k-1}}{a_k}\mathbf{v}_{k-1}$, which contradicts (2).

As an exercise, prove the remaining implication (3) $\Rightarrow$ (1). $\square$

For example, matrices

$$\mathbf{v}_1 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

are linearly dependent, because $\mathbf{v}_1 + \mathbf{v}_2 - 4\mathbf{v}_3 = \mathbf{0}$ (we use property (3)).

If $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ contains the zero vector $\mathbf{0}$, then S is linearly dependent, because we can drop $\mathbf{0}$, and the span does not change. If S contains to equal vectors, then, for the same reason, S is linearly dependent.

Example. Are the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

linearly independent?

By (3), we need to see if $x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3 = \mathbf{0}$ has only the trivial solution. The matrix of coefficients for this linear system (of x, y, z) is

$$A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}.$$

As we know, it has the only solution (trivial one) if and only if $\det(A) \neq 0$. We have $\det(A) = 18 \neq 0$. Then $x = y = z = 0$ and $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent.

This example suggests the following observation. Vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ of $\mathbb{R}^n$ are linearly independent if and only if the $n \times n$ matrix with columns $\mathbf{v}_1, \dots, \mathbf{v}_n$ has a nonzero determinant. [If the determinant is 0, then the vectors are linearly dependent.]