Lec 23: Basis and dimension

Notions of span and linear independence allow now to define basis of a vector space. Let V be a vector space. Its vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are called a *basis* of V if they are linearly independent and span V.

For example, vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis of $\mathbb{R}^3$. Indeed, they are linearly independent: if $a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ equals $\mathbf{0}$, then $a = b = c = 0$. And they span $\mathbb{R}^3$ because any vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ can be represented as the linear combination $x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$. The basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is called the *standard basis* of $\mathbb{R}^3$. Similarly, vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \ \dots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

form the *standard* basis of $\mathbb{R}^n$.

Exercises. 1°. Denote by $\mathbb{R}_n$ the vector space of all $1 \times n$ matrices. What would be the standard basis in it?

2°. Let $\mathrm{Mat}(n, m)$ be the vector space of all $n \times m$ matrices ($\mathrm{Mat}(n, 1) = \mathbb{R}^n$, $\mathrm{Mat}(1, n) = \mathbb{R}_n$). Denote by E_{ij} the matrix which (i, j) entry is 1 and all other entries are 0. Show that all E_{ij} form a basis of $\mathrm{Mat}(n, m)$.

3°. Consider the vector space $\mathrm{Pol}(n)$ of all polynomials of degree $\leq n$. Prove that the polynomials $t^n, t^{n-1}, \dots, t^2, t, 1$ form a basis in Pol_n. [It is called the *standard basis.*]

All these examples illustrate that the vectors spaces we usually consider have obvious bases. There are many other, not that obvious, vector sets which form a basis.

Example. Find out whether the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

form a basis in $\mathbb{R}^3$.

These vectors are linearly independent since the 3×3 matrix

$$A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

has the nonzero determinant (it is 1). Now check if the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ equals $\mathbb{R}^3$. For this, take an arbitrary vector $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and solve the linear equation $x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3 = \mathbf{v}$, which is equivalent to the system

$$\begin{aligned} x + y + 2z &= a, \\ 2x + y &= b, \\ y + 3z &= c, \end{aligned}$$

or $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{v}$ where matrix A is as above. Since $\det(A) \neq 0$, A is invertible. Hence the linear system has the solution $A^{-1}\mathbf{v}$. Thus we have shown that $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$. With linear independence this implies that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis of $\mathbb{R}^3$.

In this example we used only that $\det(A) \neq 0$. If $\det(A)$ was 0, the vectors would be linearly dependent by a theorem from last lecture, and therefore would not be a basis. Essentially, we proved

Theorem. *Vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ form a basis of $\mathbb{R}^n$ if and only if the determinant of the $n \times n$ matrix $A = [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \cdots \boldsymbol{v}_n]$ is nonzero.*

Note that this theorem concerns only the case of n vectors in $\mathbb{R}^n$. There is a natural question: is it possible that m vectors form a basis of $\mathbb{R}^n$ and $m \neq n$? The answer is NO, all bases consist of the same number of vectors. To illustrate it, consider the space $V = \mathbb{R}^2$. Its standard basis consists of two vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Prove that there are no bases in $\mathbb{R}^2$ consisting of one or three vectors. The former is impossible because a vector $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ can't span $\mathbb{R}^2$. Indeed, otherwise $\mathbf{e}_1 = c\mathbf{v}$ for some number c, which implies $b = 0$; but then $\mathbf{e}_2$ is not of the form $d\mathbf{v}$, hence not in the span of $\mathbf{v}$. Now consider the case of three vectors $\mathbf{u}_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} a_3 \\ b_3 \end{bmatrix}$. Of course, they may span $\mathbb{R}^2$. But it still can't be a basis since the linear independence fails (that is, there is a nontrivial linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ which equals $\mathbf{0}$). This is because the homogeneous equation $x\mathbf{u}_1 + y\mathbf{u}_2 + z\mathbf{u}_3 = \mathbf{0}$ has infinitely many solutions (its coefficient matrix is a 2×3 matrix, and there necessarily will be a free variable). This argument can be generalized to any vector space, not only $\mathbb{R}^2$. We have:

Theorem. *If $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ and $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ are bases of a vector space V, then $n = m$.*

Example. Find out whether the matrices

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$$

form a basis of $\mathrm{Mat}(2,2)$.

We know (see the exercise above) that there is a basis consisting of 4 vectors $E_{11}, E_{12}, E_{21}, E_{22}$. Then the matrices above do not constitute a basis, because their amount is 3, which is not equal to 4.

The number of elements in a basis of V is called the *dimension* of V and denoted by $\dim V$. This is a correct definition since by the theorem all bases consist of the same number of vectors. For example, $\dim R^n = \dim R_n = n$, $\dim \mathrm{Pol}(n) = n+1$, $\dim \mathrm{Mat}(m,n) = mn$. A vector space V is called *finite-dimensional*, if it has a basis consisting of finite number of vectors (and then this number equals $\dim V$). It is possible, however, that V has no such a basis, e. g. the space of all polynomials (it has a basis of infinite number of vectors: $1, t, t^2, t^3, \dots$). In this case V is called *infinite-dimensional*, and $\dim V = \infty$. We will usually deal with finite-dimensional vector spaces.

Note that any linearly independent set $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ of vectors in V can be complemented to a basis. Indeed, if $\mathrm{Span}\, S = V$, then S is already a basis. If $\mathrm{Span}\, S$ is less than V, then take a vector $\mathbf{u}_{k+1}$ not containing in $\mathrm{Span}\, S$, and add it to the set S. Now we have $S = \{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}\}$ and it is linearly independent (why?). We continue adding vectors to S until $\mathrm{Span}\, S = V$. Then we conclude that S is a basis of V. To illustrate this, consider V the space of all traceless 2×2 matrices, i. e. matrices of the form

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

and S consisting of a matrix $\mathbf{u}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Take a matrix $\mathbf{u}_2$ not in the span of S, that is not of the form $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$. Say, $\mathbf{u}_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Now the span of $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ consists of matrices $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$ for all b, c. It is not V yet, e. g. it does not contain the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Denote this matrix by $\mathbf{u}_3$ and add to S. Then $\mathrm{Span}\, S$ consist of all matrices of the form $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$. Hence $\mathrm{Span}\, S = V$ and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis of V. In particular, $\dim V = 3$.

Theorem. *Let V be a vector space of dimension n. Then any linearly independent set S of n vectors is a basis in V.*

Proof. We only need to show that $\mathrm{Span}\, S = V$. By the argument above, S can be complemented to form a basis of V. But then it will have more than n vectors, which can't be a basis. So, S is already a basis. $\square$

Example. Do polynomials

$$\mathbf{u} = 2t^2 - t + 1,\ \mathbf{v} = -t^2 + 3t + 2,\ \mathbf{w} = 5t^2 - 1$$

form a basis of $\mathrm{Pol}(2)$?

Since $\dim \mathrm{Pol}(2) = 3$ and we have three polynomials, the last theorem says, $\mathbf{u}, \mathbf{v}, \mathbf{w}$ form a basis if and only if they are linearly independent. Let's check it. For this we need to solve the equation $x\mathbf{u} + y\mathbf{v} + z\mathbf{w} = \mathbf{0}$, or $(2x - y + 5z)t^2 + (-x + 3y)t + (x + 2y - z) = \mathbf{0}$, which leads to the system

$$\begin{aligned} 2x - y + 5z &= 0, \\ -x + 3y &= 0, \\ x + 2y - z &= 0, \end{aligned} \quad .$$

The determinant of the coefficient matrix is -30, which is nonzero. Hence the system has only the trivial solution $x = y = z = 0$. Therefore, $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent and form a basis.