Lec 25: Coordinates and Isomorphisms.

[Here should be an example of finding basis for a space of solutions $A\bar{x} = \mathbf{0}$. See example on p.247 of the book.]

By means of bases all n-dimensional vector space can be identified with $\mathbb{R}^n$ as follows. Let V be an n-dimensional vector space. This means that there is a basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ of n vectors. Hence any $\mathbf{v}$ in V has a unique presentation

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n.$$

Numbers a_i are called *coordinates* of $\mathbf{v}$ in basis S. They define a vector $[\mathbf{v}]_S$ in $\mathbb{R}^n$ by the obvious formula

$$[\mathbf{v}]_S = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

Thus we have a 1-1 correspondence between V and $\mathbb{R}^n$ (this means that if $\mathbf{v} \neq \mathbf{w}$ then $[\mathbf{v}]_S \neq [\mathbf{w}]_S$). Moreover, this correspondents preserves operations. Namely, $[\mathbf{v}+\mathbf{w}]_S = [\mathbf{v}]_S + [\mathbf{w}]_S$ and $[a\mathbf{v}]_S = a[\mathbf{v}]_S$. This allows us to say that any n-dimensional vector space is essentially $\mathbb{R}^n$. Note that the basis S corresponds to the standard basis in $\mathbb{R}^n$: $[\mathbf{v}_i]_S = \mathbf{e}_i$. For example, if $V = \mathrm{Pol}(2)$ and $S = \{1, t, t^2\}$, then

$$[1 + 3t + 2t^2]_S = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \ [-t + 2t^2]_S = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

and to the sum $(1 + 3t + 2t^2) + (-t + 2t^2) = 1 - 2t + 4t^2$ it corresponds $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ which is the sum of $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$. The same for scalar multiplication. This is the identification of $\mathrm{Pol}(2)$ and $\mathbb{R}^3$. Note that the order of basis elements is important for this correspondence. Say, if $T = \{t^2, t, 1\}$, then

$$[1 + 3t + 2t^2]_T = [2t^2 + 3t + 1]_T = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

which is not equal to $[1 + 3t + 2t^2]_S$. That is why we treat S not as a basis but as *ordered* basis; so any permutation of vectors in S gives a different basis T.

Example. Let $V = \mathbb{R}^2$ and $S = \{\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$. Show that S is a basis and find $[\mathbf{v}]_S$ where $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

S is a basis because $\det(A) = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \neq 0$. Now find the coordinates of $\mathbf{v}$ in S: $x\mathbf{v}_1 + y\mathbf{v}_2 = \mathbf{v}$. This is a linear system with unknowns x, y and with coefficient matrix A. The (unique) solution is $x = 4, y = -1$ (verify!). These are the coordinates and $[\mathbf{v}]_S = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

The correspondence above suggests the following definition. A function f from a vector space V to a vector space W is called *isomorphism* if it is 1-1 correspondence and preserves operations. The above function $f\colon V \to \mathbb{R}^n$, $f(\mathbf{v}) = [\mathbf{v}]_S$ is an isomorphism. The function $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ which maps $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to $\begin{bmatrix} a+c \\ b-c \end{bmatrix}$ (e. g. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ to $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$) preserves the operations (why?) but is not 1-1 because the equation $f(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has infinitely many solutions (why?) [so, to $\mathbf{0}$ in $\mathbb{R}^2$ it corresponds many vectors in $\mathbb{R}^3$, not one]. If there is an isomorphism $f\colon V \to W$, then there is an isomorphism $f'\colon W \to V$ (why?). We say that V and W are *isomorphic*. For example Pol(2) is isomorphic to $\mathbb{R}^3$, Mat$(2, 2)$ is isomorphic to $\mathbb{R}^4$, $\mathbb{R}_n$ - to $\mathbb{R}^n$. More generally, all n-dimensional vector spaces are isomorphic to $\mathbb{R}^n$. Note that if U, V are isomorphic and V, W are isomorphic, then U, W are isomorphic (because the composition of isomorphisms is an isomorphism). Then any two n-dimensional vector spaces are isomorphic (how to construct an isomorphism?). Moreover,

Theorem. *Vector spaces of different dimensions are not isomorphic.*

Exercise: prove this theorem (hint: show first that any isomorphism takes a basis to a basis).

For example, spaces Pol(5) and Mat$(3, 3)$ are not isomorphic and Pol(5), Mat$(2, 3)$ are. Work out conditions for the space Pol(n) and Pol(k, l) to be isomorphic. Isomorphisms can be exploited to answer questions like this:

Example. Find all p for which the set $S = \{t^2 + t + 3, 4t^2 - t, 2t^2 + t + p\}$ is a not basis in Pol(2).

Under the isomorphism $f\colon$ Pol$(2) \to \mathbb{R}^3$ defined by $f(at^2+bt+c) = a\mathbf{e}_1+b\mathbf{e}_2+c\mathbf{e}_3$ the set S goes to $T = \{\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ p \end{bmatrix}\}$. Now, since isomorphisms take bases to bases, S is not a basis in Pol(2) if and only if T is not a basis in $\mathbb{R}^3$. As we know, the latter is equivalent to

$$\begin{vmatrix} 1 & 4 & 2 \\ 1 & -1 & 1 \\ 3 & 0 & p \end{vmatrix} = 0,$$

or $18 - 5p = 0$, which implies $p = 3.6$. Then S is not a basis if and only if $p = 3.6$.