Lec 26: Transition matrix.

Let V be an n-dimensional vector space and $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, $T = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ its two bases. The *transition matrix* $P_{S \leftarrow T}$ from T to S is $n \times n$ matrix which columns are coordinates of $\mathbf{w}_j$ in basis S:

$$P_{S \leftarrow T} = [[\mathbf{w}_1]_S \ [\mathbf{w}_2]_S \ \dots [\mathbf{w}_n]_S].$$

As we will see, by means of this matrix one can transform coordinates of a vector in basis T to coordinates in S. But before the theorem, let's look at examples of finding $P_{S \leftarrow T}$.

Example 1. $V = \mathbb{R}^3$, $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ — standard basis, $T = \{\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}$. Then

$$P_{S \leftarrow T} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}.$$

(e. g. $\mathbf{w}_3 = 2\mathbf{e}_2 + \mathbf{e}_3$)

Example 2. $V = \mathbb{R}_3$, $S = \{\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\}$, $T = \{\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}$. To find the coordinates x_1, x_2, x_3 of $\mathbf{w}_1$ in basis S, we have to solve the linear system:

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{w}_1.$$

Its augmented matrix is $[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 | \mathbf{w}_1]$. The RREF will be $[I_3 | \mathbf{x}]$ for some $\mathbf{x}$ (the matrix $[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$ has nonzero det, so its RREF is the identity matrix). Clearly, then $\mathbf{x}$ will be the solution. Similarly, to find the coordinates of $\mathbf{w}_2, \mathbf{w}_3$ in S, we have to solve linear systems with augmented matrices $[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 | \mathbf{w}_2]$, $[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 | \mathbf{w}_3]$. Hence we can do it at once by producing the RREF for the partitioned matrix

$$[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 | \mathbf{w}_1 | \mathbf{w}_2 | \mathbf{w}_3] = \left[\begin{array}{ccc|c|c|c} 1 & -2 & 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 & 1 \\ 3 & 0 & 1 & 0 & 2 & 1 \end{array}\right]$$

which one can find is

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1.5 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & -1 \\ 0 & 0 & 1 & -4.5 & 2 & -2 \end{array}\right]$$

. Then the last three column are exactly $[\mathbf{w}_1]_S$, $[\mathbf{w}_2]_S$ and $[\mathbf{w}_3]_S$. So, the transition matrix is

$$P_{S\leftarrow T} = \begin{bmatrix} 1.5 & 0 & 1 \\ -2 & 1 & -1 \\ -4.5 & 2 & -2 \end{bmatrix}.$$

Theorem 0.1. *For any vector $\boldsymbol{v}$ in V we have $[\boldsymbol{v}]_S = P_{S\leftarrow T}[\boldsymbol{v}]_T$.*

Proof. Let $\mathbf{v} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_n\mathbf{w}_n$ (in other words, $[\mathbf{v}]_T = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$). Then $[\mathbf{v}]_S =$

$$[c_1\mathbf{w}_1+c_2\mathbf{w}_2+\cdots+c_n\mathbf{w}_n]_S = c_1[\mathbf{w}_1]_S+c_2[\mathbf{w}_2]_S+\cdots+c_n[\mathbf{w}_n]_S = [[\mathbf{w}_1]_S\ [\mathbf{w}_2]_S]\ \ldots [\mathbf{w}_n]_S] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} =$$

$P_{S\leftarrow T}[\mathbf{v}]_T$. □

In example 2, if $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, then one can show $\mathbf{v} = 2\mathbf{w}_2 + \mathbf{w}_3$, or equivalently, $[\mathbf{v}]_T = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$. Then by Theorem we can find it's coordinates in S:

$$[\mathbf{v}]_S = \begin{bmatrix} 1.5 & 0 & 1 \\ -2 & 1 & -1 \\ -4.5 & 2 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Hence $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{v}_3$.

Point out some properties of $P_{S\leftarrow T}$:

- $P_{S\leftarrow S} = I_n$ - identity matrix (why?).
- If R, S, T are bases in V, then $P_{R\leftarrow S}P_{S\leftarrow T} = P_{R\leftarrow S}$. Indeed for any vector $\mathbf{v}$ in V we have by the Theorem: $[\mathbf{v}]_R = P_{R\leftarrow S}[\mathbf{v}]_S = P_{R\leftarrow S}(P_{S\leftarrow T}[\mathbf{v}]_T) = (P_{R\leftarrow S}P_{S\leftarrow T})[\mathbf{v}]_T$. On the other hand, we know $[\mathbf{v}]_R = P_{R\leftarrow T}[\mathbf{v}]_T$. Then $(P_{R\leftarrow S}P_{S\leftarrow T})[\mathbf{v}]_T = P_{R\leftarrow T}[\mathbf{v}]_T$. Since this holds for any $\mathbf{v}$ (hence any $[\mathbf{v}]_T$), the matrices on the left coincide (why?): $P_{R\leftarrow S}P_{S\leftarrow T} = P_{R\leftarrow S}$.
- The transition matrix from T to S is invertible and its inverse is the transition matrix from S to T: $P_{S\leftarrow T}^{-1} = P_{T\leftarrow S}$. This follows from the previous properties, if we take $R = S$.

In example 2 we could compute $P_{S\leftarrow T}$ using the properties. Denote by St the standard basis in $\mathbb{R}^3$. Then $P_{S\leftarrow T} = P_{S\leftarrow St}P_{St\leftarrow T} = P_{St\leftarrow S}^{-1}P_{St\leftarrow T}$. The transition matrices to the standard basis are obvious (example 1), so the only nontrivial thing is to find the inverse of the first matrix (do it!). We have

$$P_{S\leftarrow T} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & -5 & -0.5 \\ -1 & -1 & 1 \\ -1.5 & 3 & 2.5 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1.5 & 0 & 1 \\ -2 & 1 & -1 \\ -4.5 & 2 & -2 \end{bmatrix},$$

exactly the matrix we got before.

Example 3. Let $V = \mathrm{Pol}(1)$, $S = \{\mathbf{v}_1 = t, \mathbf{v}_2 = t-3\}$, $T = \{\mathbf{w}_1 = t-1, \mathbf{w}_2 = t+1\}$. For the standard basis $St = \{t, 1\}$ we have

$$P_{St\leftarrow S} = \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}, \quad P_{St\leftarrow T} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

As in example 2, to find $P_{S\leftarrow T}$, we have to produce RREF for the partitioned matrix

$$[P_{St\leftarrow S} | P_{St\leftarrow T}] = \begin{bmatrix} 1 & 1 & | & 1 & 1 \\ 0 & -3 & | & -1 & 1 \end{bmatrix},$$

which is

$$\begin{bmatrix} 1 & 0 & | & \frac{2}{3} & \frac{4}{3} \\ 0 & 1 & | & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}.$$

Then take the matrix on the right:

$$P_{S\leftarrow T} = \frac{1}{3} \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}.$$

Another method of finding the transition matrix is $P_{S\leftarrow T} = P^{-1}_{St\leftarrow S} P_{St\leftarrow T}$ proved before the example. Verify that this way we get the same matrix. Note that $5t - 1 = 3(t-1) + 2(t+1) = 3\mathbf{w}_1 + 2\mathbf{w}_2$. Using the theorem, find the coordinates of $5t - 1$ in basis S.