Lec 27: Rank of a matrix.

Let A be an $m \times n$ matrix. Columns of A are vectors in $\mathbb{R}^m$ and rows of A are vectors in $\mathbb{R}_n$. The spans of these vectors in $\mathbb{R}^m$ and $\mathbb{R}_n$ are called *column space* and *row space* respectively.

Example 1. The row space of matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & -1 & -1 \\ 2 & 3 & 0 & 1 \end{bmatrix}$$

is spanned by vectors $\mathbf{u}_1 = [0\ 1\ 2\ 3]$, $\mathbf{u}_2 = [1\ 1\ -1\ -1]$, $\mathbf{u}_3 = [2\ 3\ 0\ 1]$ in $\mathbb{R}_4$; the column space of A is the span of $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$.

Note that elementary row transformations do not change the row space (e. g., span of rows $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ equals the span of $\mathbf{r}_1 - c\mathbf{r}_2, \mathbf{r}_2, \mathbf{r}_3$). Hence the row spaces for A and any its REF (e. g. RREF) B are the same. For instance, matrix A in example 1 has a REF

$$B = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore the row space of A is spanned by $\mathbf{w}_1 = [1\ 1\ -1\ -1]$, $\mathbf{w}_2 = [0\ 1\ 2\ 3]$ (as well as by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$). Since $\mathbf{w}_1, \mathbf{w}_2$ are linearly independent (why?), the dimension of the row space of A is 2.

Similarly, elementary column transformations do not change column space. Then that of A is equal to the column space of its CEF C. In example 1, A has a column echelon form

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{bmatrix}.$$

We see that the column space is spanned by the first two columns. They are linearly independent, so the dimension of the column space is again 2. In fact, this is not just a coincidence:

Theorem. *Dimensions of the row space and column space are equal for any matrix A.*

[See the proof on p. 275 of the book.]

The dimension of the row space of A is called *rank* of A, and denoted $\operatorname{rank} A$. By theorem, we could define rank as the dimension of the column space of A. By above, the matrix in example 1 has rank 2. To find the rank of any matrix A, we should find its REF B, and the number of nonzero rows of B will be exactly the rank of A [another way is to find a CEF, and the number of its nonzero columns will be the rank of A].

Now make some remarks.

(1) $\operatorname{rank} I_n = n$.

(2) *Nullity* of an $m \times n$ matrix A is the dimension of the null space, i. e. the dimension of the space of solutions to $A\mathbf{x} = \mathbf{0}$. Then the nullity of A is equal to $n - \operatorname{rank} A$. Indeed, let B be a REF of A. The nullity of A is the number of columns of B without leading ones (i. e. columns corresponding to free variables). The rest columns (containing leading ones) are the basis in the column space of B, hence is equal to $\operatorname{rank} B = \operatorname{rank} A$. This proves that the nullity plus rank is the number of columns, i .e. n.

(3) An $n \times n$ matrix A is nonsingular if and only if $\operatorname{rank} A = n$ (i. e. rows of A are linearly independent). This is because nonsingular matrices are those having RREF I_n.

(4) An $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\operatorname{rank} A = \operatorname{rank} [A|\mathbf{b}]$ (i. e. ranks of coefficient and augmented matrices coincide). Indeed, for the system to have a solution, number of nonzero rows in REF of A and $[A|\mathbf{b}]$ must coincide.

Example 2. Let $\mathbf{u}_1 = t^2 + t + 1$, $\mathbf{u}_2 = t^2 - t + 2$, $\mathbf{u}_3 = 2t^2 + 3$ be vectors in Pol(2). Find a basis in $V = \operatorname{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. We use the standard isomorphism f between Pol(2) and $\mathbb{R}_3$. We have $f(\mathbf{u}_1) = [1\ 1\ 1]$, $f(\mathbf{u}_2) = [1\ -1\ 2]$, $f(\mathbf{u}_3) = [2\ 0\ 3]$. Then $f(V)$ is equal to the row space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & 3 \end{bmatrix}.$$

The RREF of A is

$$B = \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0. \end{bmatrix}$$

Hence a basis of the row space of A is $[1\ 0\ 1.5]$, $[0\ 1\ -0.5]$. Under f, it corresponds to a basis of V: $t^2 + 1.5$ and $t - 0.5$.

Another question one could possibly ask is to find a basis in V consisting of a subset of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. We already know that $\dim V = 2$, so the only thing we have to do is to pick two linearly independent vectors from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. The first two vectors are linearly independent, because the rank of matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

is 2 (why?). So, vectors $\mathbf{u}_1, \mathbf{u}_2$ form a basis in V.