Lec 31: Inner products.

An *inner product* on a vector space V assigns to vectors $\mathbf{u}, \mathbf{v}$ a real number $(\mathbf{u}, \mathbf{v})$, such that

(1) $(\mathbf{u}, \mathbf{u}) \geq 0$ for all $\mathbf{u}$, and $(\mathbf{u}, \mathbf{u}) = 0$ if and only if $\mathbf{u} = \mathbf{0}$;

(2) $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$ for all $\mathbf{u}, \mathbf{v}$;

(3) $(\mathbf{u}, a\mathbf{v} + b\mathbf{w}) = a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{w})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

As you can notice, this definition was suggested by the dot product in $\mathbb{R}^n$. Curiously, properties (4)-(7) from the previous lecture are valid for any inner product ($\mathbb{R}^n$ replaced by V), as they were obtained from the same properties (1)-(3). Thus, we have notions of length, angle, orthogonal and orthonormal sets, CBS, triangle inequalities for vectors in any vector space with inner product. The dot product is of course an inner product, and we call it the *standard inner product*. But there are much more examples of inner products.

Example 1. Let $V = \mathbb{R}^2$, $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Define

$$(\mathbf{u}, \mathbf{v}) = 2u_1v_1 - u_1v_2 - v_1u_2 + u_2v_2.$$

This is an inner product. Indeed,

$$(\mathbf{u}, \mathbf{u}) = 2u_1^2 - 2u_1u_2 + u_2^2 = u_1^2 + (u_1 - u_2)^2 \geq 0.$$

If $(\mathbf{u}, \mathbf{u}) = 0$, then from the above formula $u_1 = u_1 - u_2 = 0$, which means $\mathbf{u} = \mathbf{0}$. This implies (1). The symmetry property (2) is straightforward:

$$(\mathbf{v}, \mathbf{u}) = 2v_1u_1 - v_1u_2 - u_1v_2 + v_2u_2 = (\mathbf{u}, \mathbf{v}),$$

as well as (3):

$$(\mathbf{u}, a\mathbf{v}+b\mathbf{w}) = 2u_1(av_1+bw_1)-u_1(av_2+bw_2)-(av_1+bw_1)u_2+u_2(av_2+bw_2) = a(\mathbf{u}, \mathbf{v})+b(\mathbf{u}, \mathbf{w}).$$

The length of vector $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is $\|\mathbf{u}\| = \sqrt{(\mathbf{u}, \mathbf{u})} = \sqrt{2 \cdot 3^2 - 2 \cdot 3 \cdot 4 + 4^2} = \sqrt{10}$ (note that for the standard product we have $\|\mathbf{u}\| = \sqrt{3^2 + 4^4} = 5$). The angle φ between $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ is given by

$$\cos\varphi = \frac{(\mathbf{u}, \mathbf{v})}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{1 - \sqrt{3}}{1 \cdot \sqrt{7 - 2\sqrt{3}}}.$$

[Verify! For the standard product $\cos\varphi = 0.5$, so $\varphi = 60°$, while here $\varphi = \arccos \frac{1-\sqrt{3}}{1 \cdot \sqrt{7-2\sqrt{3}}} \approx 113°$.]

Further, when we talk about lengths and angles without specifying the inner product, we mean the standard product (hence lengths and angles are usual).

Note that the inner product in our example can be represented as (verify!)

$$(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T A \mathbf{v} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

In general, let $S = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be a basis in $\mathbb{R}^n$, and $(\cdot, \cdot)$ be an inner product. Then we have

$$(\mathbf{u}, \mathbf{v}) = (u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + \cdots + u_n\mathbf{e}_n, v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n) = \sum_{i,j=1}^{n} u_i v_j (\mathbf{e}_i, \mathbf{e}_j) = \mathbf{u}^T A \mathbf{v},$$

where A is the matrix with $a_{ij} = (\mathbf{e}_i, \mathbf{e}_j)$. Thus all inner products are of the form $(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T A \mathbf{v}$ where A is a symmetric matrix. This is called *the matrix of the inner product*. For example, the matrix of the standard (dot) product is the identity I_n.

Note that if we define a function $(\mathbf{u}, \mathbf{v})$ by this formula $\mathbf{u}^T A \mathbf{v}$ (where A is a symmetric $n \times n$ matrix), then it will not necessarily be an inner product, the properties (2) and (3) hold though. Indeed, take

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which corresponds to $(\mathbf{u}, \mathbf{v}) = u_1 v_1 - u_2 v_2$. Then $(\mathbf{u}, \mathbf{u}) = -1 < 0$ for $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and (1) fails. Then A is not a matrix of inner product. Hence the formula $(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T A \mathbf{v}$ (with symmetric A) defines an inner product in $\mathbb{R}^n$, if and only if $(\mathbf{u}, \mathbf{u}) = \mathbf{u}^T A \mathbf{u}$ is positive for all nonzero $\mathbf{u}$ (so, (1) is satisfied). Such symmetric matrices A are called *positive definite*. Thus, positive definite matrices correspond to inner products in $\mathbb{R}^n$.

Now look at other examples of inner product spaces (i. e. vector spaces with an inner product) V. Finite dimensional V with inner product are called *Euclidean spaces*. Next is an example of infinite-dimensional inner product space.

Example 2. Let V be the space of all continuous functions on the interval $[0, 1]$ (we could choose any interval $[a, b]$, $a \neq b$). Then define

$$(f, g) = \int_0^1 f(t)g(t)dt.$$

This is an inner product on V. Indeed, property (2) holds by the symmetry of multiplication of reals, and (3) follows from the properties of integral. Prove (1):

$$(f, f) = \int_0^1 f(t)^2 dt \geq 0,$$

because $f(t)^2 \geq 0$. Moreover, $(f, f) = 0$ clearly implies f is a zero function. CBS inequality in this case is:

$$\left| \int_0^1 f(t)g(t)dt \right| \leq \sqrt{\int_0^1 f(t)^2 dt} \sqrt{\int_0^1 g(t)^2 dt},$$

or, after squaring,

$$(\int_0^1 f(t)g(t)dt))^2 \leq \int_0^1 f(t)^2 dt \int_0^1 g(t)^2 dt.$$

As an exercise, write down the triangle inequality for this inner product.

Note that

$$(\cos 2\pi t, 1) = \int_0^1 \cos 2\pi t dt = 0,$$

so the functions $\cos 2\pi t$ and 1 are orthogonal. Moreover, one can show that the infinite set

$$1,\ \cos 2\pi t,\ \sin 2\pi t,\ \cos 4\pi t,\ \sin 4\pi t,\ \ldots,\ \cos 2n\pi t,\ \sin 2n\pi t,\ \ldots$$

is orthogonal. Therefore, by (6) (see previous lecture), any its finite subset is linearly independent.

Example 3. Consider the same inner product but on the space $V = \text{Pol}(2)$. For polynomials $p(t)$, $q(t)$ we have

$$(p(t), q(t)) = \int_0^1 p(t)q(t)dt.$$

Find an orthogonal basis $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in V. Take $\mathbf{e}_1 = 1$. Now find $\mathbf{e}_2$ in the form $at+b$. It must be orthogonal to $\mathbf{e}_1$, so $\int_0^1 at + bdt = 0$, which implies $\frac{a}{2} + b = 0$. Then we can take $\mathbf{e}_2 = 2t-1$. Finally, find $\mathbf{e}_3$ in the form at^2+bt+c. It must be orthogonal to e_1 and e_2. Solving corresponding equations, we find out that the vector $6t^2-6t+1$ suits for $\mathbf{e}_3$. Since S is orthogonal set, it is linearly independent, and hence a basis in V (because $\dim V = 3$). To make it orthonormal, we need to replace all $\mathbf{e}_i$ by $a_i\mathbf{e}_i$ so that $(a_i\mathbf{e}_i, a_i\mathbf{e}_1) = 1$. For instance, we can take $a_1 = 1$, because $(\mathbf{e}_1, \mathbf{e}_1) = 1$ already. But

$$(\mathbf{e}_2, \mathbf{e}_2) = \int_0^1 (2t-1)^2 dt = \int_0^1 (4t^2 - 4t + 1) = (\frac{4t^3}{3} - 2t^2 + t)|_0^1 = \frac{1}{3},$$

so we take $a_2 = \sqrt{3}$. As an exercise, find a_3.