Lec 33: Orthogonal complements and projections.

Let S be a set of vectors in an inner product space V. The *orthogonal complement* $S^\perp$ to S is the set of vectors in V orthogonal to all vectors in S. The orthogonal complement to the vector $\begin{bmatrix}1\\2\\3\end{bmatrix}$ in $\mathbb{R}^3$ is the set of all $\begin{bmatrix}x\\y\\z\end{bmatrix}$ such that $x+2x+3z=0$, i. e. a plane. The set $S^\perp$ is a subspace in V: if $\mathbf{u}$ and $\mathbf{v}$ are in $S^\perp$, then $a\mathbf{u}+b\mathbf{v}$ is in $S^\perp$ for any reals a, b. Indeed, for a $\mathbf{w}$ in S we have $(a\mathbf{u}+b\mathbf{v}, \mathbf{w}) = a(\mathbf{u}, \mathbf{w}) + b(\mathbf{v}, \mathbf{w}) = 0$ since $(\mathbf{u}, \mathbf{w}) = (\mathbf{v}, \mathbf{w}) = 0$. Similarly, if a vector is orthogonal to S, it is orthogonal to $W = \operatorname{Span} S$: $S^\perp = W^\perp$.

Example 1. Let $V = \mathbb{R}_4$, $W = \operatorname{Span}\{\mathbf{u} = [1\ 1\ 0\ 2], \mathbf{v} = [2\ 0\ 1\ 1], \mathbf{w} = [1\ -1\ 1\ -1]\}$. Find a basis of $W^\perp$.

By the previous, $W^\perp$ consist of vectors $\mathbf{x} = [x_1\ x_2\ x_3\ x_4]$ such that $(\mathbf{x}, \mathbf{u}) = (\mathbf{x}, \mathbf{v}) = (\mathbf{x}, \mathbf{w}) = 0$, i. e.

$$\begin{aligned} x_1 + x_2 + 2x_4 &= 0 \\ 2x_1 + x_3 + x_4 &= 0 \\ x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}.$$

In other words, $W^\perp$ is the nullspace of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

Its RREF is

$$B = \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Putting r, s free and $x_3 = 2r$, $x_4 = 2s$, we have

$$\mathbf{x} = [-1\ 1\ 2\ 0]r + [-1\ -3\ 0\ 2]s,$$

hence vectors $[-1\ 1\ 2\ 0]$ and $[-1\ -3\ 0\ 2]$ form a basis in $W^\perp$.

An important property of orthogonal complements is related with the notion of direct sum. If U and W are subspaces in V, then by $U+W$ we denote the span of U and W. Therefore $U+W$ is a subspace in V, and its elements can be written as $\mathbf{u}+\mathbf{w}$ for all $\mathbf{u}$ in U and $\mathbf{w}$ in W. In particular, $U+U=U$. If $V=\mathbb{R}_4$, U is the subspace of all vectors of the form $[a\ b\ 0\ 0]$, W the subspace of all $[0\ c\ d\ 0]$, then $U+W$ consists of all vectors $[x\ y\ z\ 0]$. In this example U and W intersect by a 1-dimensional subspace. If we take $W = \{[0\ 0\ c\ d]\}$, then $U+W=V$ and the intersection $U\cap W$ consist only of $\mathbf{0}$.

We say that V is the *direct sum* of its subspaces U and W if $V = U+W$ and $U \cap W = \{\mathbf{0}\}$. In this case we write $V = U \oplus W$. As an exercise, show that $\dim U + \dim W = \dim V$. For example, $\operatorname{Pol}(2) = \operatorname{Pol}(1) \oplus \operatorname{Span}\{t^2\} = \operatorname{Span}\{1\} \oplus \operatorname{Span}\{t\} \oplus \operatorname{Span}\{t^2\}$.

Theorem. *For any subspace U in an inner product space V holds $V = U \oplus U^\perp$.*

Proof. Show that $U \cap U^\perp = \{\mathbf{0}\}$. If $\mathbf{u}$ belongs to both U and $U^\perp$, then $(\mathbf{u}, \mathbf{u}) = 0$ which implies $\mathbf{u} = \mathbf{0}$ by the definition of inner product.

Now denote $W = U + U^\perp$ and prove $W = V$. We can choose an orthonormal basis in W and extend it to orthonormal basis in V. Thus, if $W \neq V$, there is an element $\mathbf{e}$ in the basis of V orthogonal to W. Since W contains U, $\mathbf{e}$ is orthogonal to U as well, which means $\mathbf{e}$ belongs to $U^\perp$. The latter is a subspace of W, therefore $\mathbf{e}$ is in W, and we arrive at contradiction with choice of $\mathbf{e}$. $\square$

By the theorem, any vector $\mathbf{v}$ in V can be uniquely (why?) represented as $\mathbf{u} + \mathbf{w}$, where $\mathbf{u} \in U$, $\mathbf{w} \in U^\perp$. Then vector $\mathbf{u}$ is called the *projection* of $\mathbf{v}$ onto U, and we denote it by $\operatorname{proj}_U \mathbf{v}$. In $V = \mathbb{R}^2$ regarded as plane, let U be the x-line. Then $U^\perp$ is y-line and the projection of (x, y) on U is $(x, 0)$.

Projection can also be characterized by the following property: $\mathbf{u} = \operatorname{proj}_U \mathbf{v}$ is the closest to $\mathbf{v}$ vector in U. By being closest we mean the distance $\|\mathbf{u} - \mathbf{v}\|$ is as small as possible. [For the proof see the theorem on p.343 of the book.] We call $\|\mathbf{v} - \operatorname{proj}_U \mathbf{v}\|$ the distance between $\mathbf{v}$ and U. Given an orthonormal basis $S = \{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ in U, we have $\mathbf{u} = (\mathbf{u}, \mathbf{e}_1)\mathbf{e}_1 + \cdots + (\mathbf{u}, \mathbf{e}_k)\mathbf{e}_k$. Now note that $(\mathbf{u}, \mathbf{e}_i) = (\mathbf{v}, \mathbf{e}_i)$ since $\mathbf{v} = \mathbf{u} + \mathbf{w}$ and $\mathbf{w}$ is orthogonal to U. Thus we have a simple formula for computing the projection:

$$\operatorname{proj}_U \mathbf{v} = (\mathbf{v}, \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{v}, \mathbf{e}_2)\mathbf{e}_2 + \cdots + (\mathbf{v}, \mathbf{e}_k)\mathbf{e}_k.$$

Example 2. Let $V = \mathbb{R}^3$, U the orthogonal complement to $\begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$.

(a) Find a basis of U;

(b) Find an orthonormal basis of U;

(c) Find the distance between $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$ and U.

Subspace U consists of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that

$$x + 2y - 5z = 0.$$

Setting $y = r$, $z = s$ free, we have $x = -2r + 5s$ and

$$\mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} r + \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} s.$$

Hence vectors $\mathbf{u}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$ are a basis of U.

Now find an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of U. Take $\mathbf{e}_1 = \frac{1}{\|\mathbf{u}_1\|}\mathbf{u}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ to satisfy $\|\mathbf{e}_1\| = 1$. Let's find $\mathbf{e}_2$ in the form $a\mathbf{u}_1 + b\mathbf{u}_2$. Coefficients a, b can be found from the conditions we want from $\mathbf{e}_2$: $(\mathbf{e}_2, \mathbf{e}_1) = 0$ (or equivalently $(\mathbf{e}_2, \mathbf{u}_1) = 0$) and $(\mathbf{e}_2, \mathbf{e}_2) = 0$. Solving this (do it!) we find $\mathbf{e}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ (or the negative). This finishes (b).

In order to find the distance form $\mathbf{v}$ to U, we have to find the projection $\mathbf{u} = \mathrm{proj}_U\, \mathbf{v}$. As soon as we've found an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ in U, we can use the formula:

$$\mathbf{u} = (\mathbf{v}, \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{v}, \mathbf{e}_2)\mathbf{e}_2 = -\sqrt{5}\mathbf{e}_1 + 2\sqrt{6}\mathbf{e}_2 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}.$$

We have $\mathbf{w} = \mathbf{v} - \mathbf{u} = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$ and the distance

$$\|\mathbf{v} - \mathbf{u}\| = \|\mathbf{w}\| = \sqrt{30}.$$

Exercise Let $V = \mathrm{Pol}(2)$ with the inner product $(p, q) = \int_0^1 p(t)q(t)dt$. Find the approximation of t^2 by a linear polynomial. [In other words, find the projection of t^2 on the subspace Pol(1).] (Answer: $t - \frac{1}{6}$) What is the distance between t^2 and Pol(1)?

Homework # 6 (due 23 November)

1. Section 5.3 p. 317 Exercise 3.
2. Section 5.3 p. 319 Exercise 36(a).
3. Section 5.3 p. 319 Exercise 45. (See p.311 for the definition of positive definite).
4. Section 5.5 p. 348 Exercise 7.
5. Section 5.5 p. 349 Exercise 16.
6. Section 6.1 p.372 Exercise 1.
7. Section 6.2 p.387 Exercise 3.