Lec 34: Linear transformations

Linear transformation is a more general version of a matrix transformation. Recall that the latter is a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ represented by a $m \times n$ matrix A, namely $f(\mathbf{x}) = A\mathbf{x}$ (vector $\mathbf{x}$ belongs to $\mathbb{R}^n$, hence $A\mathbf{x}$ lies in $\mathbb{R}^m$). Note that f is linear in the following sense: $f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ and $f(c(\mathbf{x})) = cf(\mathbf{x})$. [Indeed, $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x}+A\mathbf{y}$ and $A(c\mathbf{x}) = cA\mathbf{x}$.] Now let V and W be vector spaces. A *linear transformation* of V into W is a function $L\colon V \to W$ such that $L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x})+L(\mathbf{y})$ and $L(c\mathbf{x}) = cL(\mathbf{x})$ for all reals c and vectors $\mathbf{x}, \mathbf{y}$. Equivalently, $L(a\mathbf{x} + b\mathbf{y}) = aL(\mathbf{x}) + bL(\mathbf{y})$, for all reals a, b and vectors $\mathbf{x}, \mathbf{y}$ in V. When we write $\mathbf{x}+\mathbf{y}$, we mean $+$ in V, and for $L(\mathbf{x}) + L(\mathbf{y})$ the $+$ is in W. Linear transformations $L\colon V \to V$ (i. e. when $W = V$) are called also *linear operators* on V.

Examples of linear transformations (LT):

(1) Matrix transformations $L(\mathbf{x}) = A\mathbf{x}$. Here $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, A is an $m \times n$ matrix. As we will see, any linear transformation $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation.

(2) Let W be the vector space of all functions $f\colon \mathbb{R} \to \mathbb{R}$ and V its subspace of all differentiable functions. Then $L\colon V \to W$ defined by $L(f) = f'$ is a linear transformation, because $L(af + bg) = (af + bg)' = af' + bg' = aL(f) + bL(g)$.

(3) Let $V = C[a, b]$ be the space of all continuous functions on the interval $[a, b]$. Then $L\colon V \to \mathbb{R}$, $L(f) = \int_a^b f(x)dx$, is a linear transformation of V into $\mathbb{R}$.

Any LT takes $\mathbf{0}$ to $\mathbf{0}$: $L(\mathbf{0}) = \mathbf{0}$. In fact, $L(\mathbf{0}) = L(0 \cdot \mathbf{0}) = 0 \cdot L(\mathbf{0}) = \mathbf{0}$. For example, $L\colon \mathbb{R} \to \mathbb{R}_2$, $L(x) = [x + 1\ 2x]$, is not LT, since $L(0) = [1\ 0] \neq [0\ 0]$. The function $L\colon \mathbb{R} \to \mathbb{R}$, $L(x) = x^2$, is not linear: $L(x+y) = (x+y)^2$, $L(x)+L(y) = x^2+y^2$, hence $L(x + y) \neq L(x) + L(y)$ in general (take, e. g., $x = y = 1$).

Due to linearity, any LT $L\colon V \to W$ is completely determined by the images of vectors from a basis in V. For example, let $L\colon \mathbb{R}^2 \to \mathbb{R}$ be a LT, and it is known that $L(\mathbf{e}_1) = 2$, $L(\mathbf{e}_2) = 1$, where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a standard basis in $\mathbb{R}^2$. Then we can recover L. Say, $L(\begin{bmatrix} 2 \\ -1 \end{bmatrix}) = L(2\mathbf{e}_1 - \mathbf{e}_2) = 2L(\mathbf{e}_1) - L(\mathbf{e}_2) = 2 \cdot 2 - 1 = 3$. Similarly, let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be a LT, and $L(\mathbf{e}_1) = \mathbf{w}_1, L(\mathbf{e}_2) = \mathbf{w}_2, \dots, L(\mathbf{e}_n) = \mathbf{w}_n$. Then for any $\mathbf{x} = a_1\mathbf{e}_1 + a_2\mathbf{w}_2 + \cdots + a_n\mathbf{e}_n$ one has $L(\mathbf{x}) = a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \cdots + a_n\mathbf{w}_n$. In other words, if we denote $m \times n$ matrix $[\mathbf{w}_1\ \mathbf{w}_2\ \dots\ \mathbf{w}_n]$ by A, then $L(\mathbf{x}) = A\mathbf{x}$. So, any LT $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation. The corresponding matrix A is called the *standard matrix representing* L. The word 'standard' here is to indicate that the coordinates of vectors are taken in the standard bases.

Example 1. Let $L\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the LT defined by

$$L(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x + 2z \\ 3z - y \end{bmatrix}.$$

Find the standard matrix A representing L. We have $L(\mathbf{e}_1) = L(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $L(\mathbf{e}_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $L(\mathbf{e}_3) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Hence

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}.$$