**Transformation of a matrix to a row echelon form**

**Example.** Using elementary row transformations, produce a row echelon form $A'$ of the matrix

$$A = \begin{bmatrix} 0 & 2 & 8 & -7 \\ 2 & -2 & 4 & 0 \\ -3 & 4 & -2 & -5 \end{bmatrix}.$$

We know that the first nonzero column of $A'$ must be of view $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. We can't achieve this from matrix $A$ unless interchange the first row with a row having a nonzero number in the first place. So let's transpose first and second rows:

$$B = A_{r_1 \leftrightarrow r_2} = \begin{bmatrix} 2 & -2 & 4 & 0 \\ 0 & 2 & 8 & -7 \\ -3 & 4 & -2 & -5 \end{bmatrix}.$$

Multiply the first row by $\frac{1}{2}$ so that the first entry becomes 1:

$$C = B_{2r_1 \to r_2} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 2 & 8 & -7 \\ -3 & 4 & -2 & -5 \end{bmatrix}.$$

Now add 3 times row 1 of $C$ to its third row to get a matrix with the first column of desired form:

$$D = C_{3r_1 + r_3 \to r_3} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 2 & 8 & -7 \\ 0 & 1 & 4 & -5 \end{bmatrix}.$$

Now we can forget about the first row and concentrate on submatrix of $D$ consisting of the other rows. Again, we need to transform this submatrix to that with the first nonzero column $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. To this end, we procede in a similar way. Now we don't need to interchange second and third rows: just multiply the second one by $\frac{1}{2}$ and then subtract it from row 3, and get the desired view of the second column. But in order to avoid fractions, let's swap rows 2 and 3 and then replace row 3 by its sum with a multiple of row 2.

$$E = D_{r_2 \leftrightarrow r_3} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 4 & -5 \\ 0 & 2 & 8 & -7 \end{bmatrix}$$

and

$$F = E_{-2r_2+r_3\to r_3} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

It is left to multiply the last row by $\frac{1}{3}$:

$$G = F_{3r_3\to r_3} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus we obtained a matrix $A' = G$ in a row echelon form.

This example suggests a general way to produce a row echelon form of an arbitrary matrix $A = [a_{ij}]$. We find the first nonzero column (*pivot column* of $A$) and the first nonzero entry in it (it is called *pivot*; in the previous example, pivot of $A$ is $a_{21} = 2$). Let the pivot be $a_{ij}$ for some $i, j$. Interchange rows 1 and $i$ getting matrix $B = [b_{ij}]$ with the pivot $b_{1j}$. Multiply the first row of $B$ by $\frac{1}{b_{1j}}$ and obtain a matrix $C$ with the pivot $c_{1j} = 1$. Then add multiples of row 1 of $C$ to the rest of rows, getting matrix $D$ with the pivot column of view $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Now focus on the submatrix of $D$ consisting of all rows except the first. Proceding in the same way with this submatrix we will finally arrive at a matrix $A'$ having row echelon form.

In general, matrix $A'$ is not uniquely determined by $A$. In the previous example, if we add a multiple of the last row of $A' = G$ to its second row, we get another echelon form of $A$. If we add 5 times row 3 of $G$ to row 2, we will have

$$H = G_{r_2+5r_3\to r_2} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now let's add the second row to the first one:

$$I = H_{r_1+r_2\to r_1} = \begin{bmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have matrix $I$ in a *reduced echelon form*, i. e. an echelon form plus the condition that all entries above leading ones are 0. Like above, any matrix can be transformed to that in a reduced echelon form. Unlike echelon form, reduced echelon form is unique for any matrix. In other words, if matrices $A'$ and $A''$ are obtained from $A$ by a sequences of elementary row transformations, and both $A', A''$ are in a reduced echelon form, then $A' = A''$. This is a theorem which needs a proof, but we won't give it now.