Lec 9: Solving Linear Systems

Solving linear system $A\bar{x} = \bar{b}$ can be carried out in two steps:

- Producing a row echelon form $[A'|\bar{b}']$ of the augmented matrix $[A|\bar{b}]$.
- Solving the linear system $A'\bar{x} = \bar{b}'$ by back substitution.

This method is called *Gaussian elimination*. If in the first step one gets the reduced row echelon form of $[A|\bar{b}]$, one uses *Gauss-Jordan reduction* method.

Example. Solve the linear system

$$\begin{aligned} x - y + 2z &= 0 \\ 3x - z &= 1 \\ -2x + 4y + z &= -2 \end{aligned} \tag{1}$$

We will use the Gaussian elimination. That is, we will transform the augmented matrix

$$B = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 0 & -1 & 1 \\ -2 & 4 & 1 & -2 \end{bmatrix}.$$

to a row echelon form. We have:

$$C = B_{r_2 - 3r_1 \to r_2} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 3 & -7 & 1 \\ -2 & 4 & 1 & -2 \end{bmatrix}, \qquad D = C_{r_3 + 2r_1 \to r_3} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 3 & -7 & 1 \\ 0 & 2 & 5 & -2 \end{bmatrix},$$

$$E = D_{3r_2 \to r_2} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -\frac{7}{3} & \frac{1}{3} \\ 0 & 2 & 5 & -2 \end{bmatrix}, \qquad F = E_{r_3 - 2r_1 \to r_3} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -\frac{7}{3} & \frac{1}{3} \\ 0 & 0 & \frac{29}{3} & -\frac{8}{3} \end{bmatrix},$$

$$G = F_{\frac{29}{3} r_3 \to r_3} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -\frac{7}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{8}{29} \end{bmatrix}.$$

Matrix G has a row echelon form. Now we have to solve the linear system with the augmented matrix G:

$$\begin{aligned} x - y + 2z &= 0 \\ y - \tfrac{7}{3}z &= \tfrac{1}{3} \\ z &= -\tfrac{8}{29} \end{aligned} \tag{2}$$

The bottom equation tells us what z is. The second one yields $y = \frac{7}{3}z + \frac{1}{3} = \frac{7}{3}(-\frac{8}{29}) + \frac{1}{3} = -\frac{9}{29}$. Finally, from the top equation we have $x = y - 2z = -\frac{9}{29} - 2(-\frac{8}{29}) = \frac{7}{29}$.

Thus, system (1) has a unique solution $x = \frac{7}{29}$, $y = -\frac{9}{29}$, $z = -\frac{8}{29}$.

One may wonder why the solutions of systems (1) and (2) are the same. Note that matrix G is produced from matrix B by a sequence of elementary row transformations. These transformations correspond to: (i) interchanging equations; (ii) multiplication of an equation by a nonzero scalar; (iii) adding a multiple of an equation to another one. Each of these operations transforms a linear system to an equivalent one (i.e. with the same solutions). Hence so does a sequence of such operations. This explains why the system (1) with matrix B has the same solutions as the system (2) with matrix G.

In our example we could have proceeded transforming G to a reduced echelon form. [That is solving system (1) by the method of Gauss-Jordan reduction.] We have:

$$H = G_{r_2+\frac{7}{3}r_3\to r_2} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & -\frac{9}{29} \\ 0 & 0 & 1 & -\frac{8}{29} \end{bmatrix}, \qquad I = G_{r_1-2r_3\to r_1} = \begin{bmatrix} 1 & -1 & 0 & \frac{16}{29} \\ 0 & 1 & 0 & -\frac{9}{29} \\ 0 & 0 & 1 & -\frac{8}{29} \end{bmatrix},$$

$$J = I_{r_1+r_2\to r_1} = \begin{bmatrix} 1 & 0 & 0 & \frac{7}{29} \\ 0 & 1 & 0 & -\frac{9}{29} \\ 0 & 0 & 1 & -\frac{8}{29} \end{bmatrix}.$$

Matrix J is in a reduced row echelon form, and the corresponding linear system is:

$$\begin{aligned} x &= \tfrac{7}{29} \\ y &= -\tfrac{9}{29} \\ z &= -\tfrac{8}{29} \end{aligned} \tag{3}$$

This system is trivial, and we don't need to perform any other operations. In fact, this is a general observation. In Gauss-Jordan method we don't need to solve a system by the back substitution because we were solving it implicitly when getting matrices from an echelon form to a reduced echelon form. For example, a linear system with an augmented matrix

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 5 \\ 0 & 1 & 4 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \tag{4}$$

[which is in a reduced row echelon form] has the solution $x_1 = 5 - 3x_3$, $x_2 = 1 - 4x_3$, $x_4 = 2$. [If the variables are x_1, x_2, x_3, x_4.] Here x_3 can be any number, so the system has infinitely many solutions. If the bottom row of matrix (4) was [0 0 0 0 2], then the system would be inconsistent (i.e. no solutions). Generally, given a system with an augmented matrix $B = [A|\bar{b}]$ in a reduced row echelon form, to get solution, we have to express variables corresponding to leading ones of B through other variables and entries of vector $\bar{b}$.