Triads and short SO_3-subgroups of compact groups

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Let G be a compact Lie group, let $\mathfrak{g} = \operatorname{Lie} G$, let $\operatorname{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation, and let $\operatorname{ad}\colon \mathfrak{g} \to \mathfrak{gl}(g)$ be the differential of Ad.

By a *triad* in G we mean a family of three conjugate involutive elements with product the identity (in particular, they commute). For instance, the elements $\operatorname{diag}(1,-1,-1)$, $\operatorname{diag}(-1,1,-1)$, and $\operatorname{diag}(-1,-1,1)$ form a triad in the group SO_3 (and any other triad is conjugate to this one). A subgroup $H \subset G$ isomorphic to SO_3 is called a *short* SO_3*-subgroup* if the dimension of any irreducible $\operatorname{Ad} H$-submodule of $\mathfrak{g}$ does not exceed 5 (that is, is equal to 1, 3, or 5).

Everywhere below, G stands for a connected simple compact Lie group with trivial centre. In [1] all triads (up to conjugation) and short SO_3-subgroups of G were found, and the following theorem was proved using the coincidence of the lists obtained.

Theorem 1. *Every triad in the group G is contained in a unique* (*up to conjugation*) *short* SO_3*-subgroup.*

É. B. Vinberg suggested that one can find a proof of the theorem more conceptual than comparison of classifications. The aim of the present note is to give such a proof. In what follows, we use Lie group theory [2] together with the theory of Riemannian and symmetric spaces [3], [4].

Let X be the symmetric space of the group G. It is clear that the action $G : X$ is effective. We denote by $s_x \in \operatorname{Isom} X$ the symmetry with respect to the point $x \in X$.

Lemma 1. *Let $\gamma\colon \mathbb{R} \to X$ be a geodesic and let $P\colon \mathbb{R} \to G$ be the group of* (*parallel*) *shifts along γ: $P(t)\gamma(0) = \gamma(t)$. Suppose that the points $x, y \in \operatorname{Im}\gamma$ are such that $s_x s_y = s_y s_x$. Let $s = s_x s_y$ and denote by $p \in \operatorname{Im} P$ the shift from x to y. Then*

1) *$P(t) = \exp t\xi$, where $\xi \in \mathfrak{g}$, $s_x\xi = s_y\xi = -\xi$, and $s\xi = \xi$,*
2) *$p^2 = s$* (*in particular, $py = x$, and thus the curve γ is closed*).

A geodesic $\gamma\colon \mathbb{R} \to X$ is said to be *shortest with respect to the points* $x, y \in \operatorname{Im}\gamma$ if it contains the shortest segment with endpoints x and y.

Lemma 2. *In the notation of Lemma 1, let $\mathfrak{m} = \{\eta \in \mathfrak{g} \mid s\eta = -\eta\}$. In this case if γ is a shortest geodesic with respect to the points x of y and has period π, then* $\operatorname{ad}\xi|_{\mathfrak{m}}^2 = -1$.

Proof. We can assume that $\gamma(0) = x$. Then $\gamma(\pi/2) = y$ and the points x and $\gamma(t)$ are not conjugate along the geodesic γ for $t \in (0, \pi/2)$. According to [4] (Chap. 7, Proposition 3.1), the points x and $\gamma(t)$ are conjugate along γ if and only if among the non-zero eigenvalues of the operator $\operatorname{ad} t\xi : \mathfrak{g}(\mathbb{C})$ there is a multiple of πi. Since $\exp \pi\xi = s$, it follows that all the eigenvalues of the operator $\operatorname{ad}\xi|_{\mathfrak{m}(\mathbb{C})}$ have the form $(2k+1)i$, $k \in \mathbb{Z}$. This, together with the absence of conjugate points for $t \in (0, \pi/2)$, implies that all the eigenvalues of this operator are equal to $\pm i$ and completes the proof of Lemma 2.

Suppose now that $S = \{s_1, s_2, s_3\}$ is a triad in G. In what follows, the index r ranges over the numbers $1, 2, 3$. We write $K_r = Z(s_r)$, $L = Z(S)$, $\mathfrak{k}_r = \operatorname{Lie} K_r$, and $\mathfrak{l} = \operatorname{Lie} L$. The triad S naturally induces a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-grading on the algebra $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{j}_1 \oplus \mathfrak{j}_2 \oplus \mathfrak{j}_3,$$

AMS 2000 *Mathematics Subject Classification.* Primary 22C05; Secondary 22E20.
DOI 10.1070/RM2007v062n05ABEH004465.

where $\mathfrak{j}_r$ stands for the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{k}_r$. We denote the orthogonal complement of $\mathfrak{k}_r$ in $\mathfrak{g}$ by $\mathfrak{m}_r$. Since the Lie algebra $\mathfrak{g}$ is simple, we have $[\mathfrak{m}_r, \mathfrak{m}_r] = \mathfrak{k}_r$.

Let us consider the symmetric space $X = G/K_1$. By the definition of K_1, every symmetry on X has exactly one fixed point. Let $o_r \in X$ be a point such that $s_{o_r} = s_r$. We denote the triple $\{o_1, o_2, o_3\}$ by S_X.

An element $\xi \in \mathfrak{j}_r$ is called an *S-characteristic* element if $\operatorname{ad}\xi|^2_{\mathfrak{m}_r} = -1$. In this case $\exp \pi\xi = s_r$. We recall that by a *defining vector* of an $\mathfrak{sl}_2$-subalgebra (of a complex Lie algebra) one means any semisimple element of it which can be supplemented by two nilpotent elements to form a standard $\mathfrak{sl}_2$-triple. Two $\mathfrak{sl}_2$-subalgebras of $\mathfrak{g}(\mathbb{C})$ are conjugate if and only if their defining vectors are conjugate ([5], Theorem 8.1).

Lemma 3. *Let $\xi_1 \in \mathfrak{j}_1$ and $\xi_2 \in \mathfrak{j}_2$ be S-characteristic elements and let $[\xi_1, \xi_2] = \xi_3 \in \mathfrak{j}_3$. Then the vectors ξ_1, ξ_2, ξ_3 form a basis of the tangent algebra of a short* SO_3*-subgroup containing the triad S. Any two vectors in the family $\{\xi_1, \xi_2, \xi_3\}$ are conjugate by an element of $N(S)$.*

Proof. One can readily prove that $[\xi_3, \xi_1] = \xi_2$ and $[\xi_2, \xi_3] = \xi_1$. Thus, the indicated vectors span a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ isomorphic to $\mathfrak{so}_3$. Let $\mathfrak{h} = \operatorname{Lie} H$, where $H \subset G$ is a connected subgroup. Since $[\mathfrak{m}_r, \mathfrak{m}_r] = \mathfrak{k}_r$, we see that the eigenvalues of the operator $\operatorname{ad}\xi_1$ are equal to 0, $\pm i$, and possibly $\pm 2i$. Hence, any defining vector of the subalgebra $\mathfrak{h}(\mathbb{C})$ (for instance, $2i\xi_1$) has the eigenvalues 0, ± 2, and possibly ± 4. Thus, $H \subset G$ is a short SO_3-subgroup. It contains the elements $s_1 = \exp \pi\xi_1$ and $s_2 = \exp \pi\xi_2$, and thus the entire triad S. The rest of the proof is obvious.

A geodesic γ on X is said to be *S-shortest* if γ is shortest with respect to some pair of points in S_X.

Lemma 4. *Let $\xi \in \mathfrak{j}_r$. The following conditions are equivalent*:

1) *ξ is S-characteristic*;
2) *ξ is the velocity vector of an S-shortest geodesic of period π*;
3) *$2i\xi$ is a defining vector of the tangent algebra of the complexification of a short* SO_3*-subgroup of G containing the triad S.*

All S-characteristic elements are $N(S)$-equivalent.

Proof. We can assume that $\xi \in \mathfrak{j}_1$. Let the element ξ be S-characteristic. We take the velocity vector $\eta \in \mathfrak{j}_2$ of any geodesic that is shortest with respect to the points o_1 and o_3 and has period π. Then by Lemma 2 the vector η is S-characteristic, and by Lemma 3 the vectors ξ and η are conjugate by an element of the group K_3. Thus, ξ is the velocity vector of some geodesic that is shortest with respect to o_2 and o_3 and has period π. This proves the implication 1) $\Rightarrow$ 2). The implications 2) $\Rightarrow$ 3) $\Rightarrow$ 1) are obvious.

Let us prove the last assertion. Let ξ and η be S-characteristic elements. There is an element $g \in N(S)$ such that the vectors ξ and $(\operatorname{Ad} g)\eta$ belong to distinct homogeneous components with respect to the above $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-grading. By Lemma 3, these vectors are $N(S)$-conjugate. Thus, the same holds for the vectors ξ and η. This completes the proof of Lemma 4.

The proof of the main theorem is now obvious. The existence follows immediately from Lemmas 2 and 3, and the uniqueness follows from Lemma 4 and Theorem 8.1 in [5].

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Presented by È. B. Vinberg
Accepted 31/JUL/07
Translated by A. I. SHTERN