Calculus Review and Vectors in 2D (13.1)

Name: Solutions

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Summary of the Sections

(1) Components of $\mathbf{v}=\overrightarrow{PQ}$, where $P=(a_1,b_1)$ and $Q=(a_2,b_2)$ are: $\mathbf{v}=\langle a_2-a_1, b_2-b_1\rangle$.

(2) The length $\|\mathbf{v}\|$ of $\mathbf{v}=\langle a,b\rangle$ is equal to $\boxed{\sqrt{a^2+b^2}}$ (1).

(3) *Vector addition:* $\langle v_1,v_2\rangle+\langle w_1,w_2\rangle = \boxed{\langle v_1+w_1, v_2+w_2\rangle}$ (2).

(4) *Scalar multiplication:* $\|\lambda\mathbf{v}\|=|\lambda|\|\mathbf{v}\|$ for λ real.

(5) $\mathbf{v}$ and $\mathbf{w}$ are *parallel* if, for some scalar λ, $\boxed{\mathbf{w}=\lambda\mathbf{v}}$ (3).

(6) If $\mathbf{v}=\langle v_1,v_2\rangle$ makes an angle θ with the positive x-axis, then $v_1=\|\mathbf{v}\|\cos\theta$ and $v_2=\|\mathbf{v}\|\sin\theta$.

Problems

Vectors in 2D

(1) Let $R=(-2,7)$. Calculate the following:

(a) The length of $\overrightarrow{OR}$.
Solution: $\sqrt{53}$.

(b) The components of $\mathbf{u}=\overrightarrow{PR}$, where $P=(1,2)$.
Solution: $\langle -3,5\rangle$.

(c) The point P such that $\overrightarrow{PR}$ has components $\langle -2,7\rangle$.
Solution: $P=(0,0)$.

(d) The point Q such that $\overrightarrow{RQ}$ has components $\langle 8,-3\rangle$.
Solution: $Q=(6,4)$

(2) Find the vector:

(a) Unit vector $\mathbf{e}_{\mathbf{v}}$ where $\mathbf{v}=\langle 3,4\rangle$.
Solution: $\langle \frac{3}{5},\frac{4}{5}\rangle$.

(b) Vector of length 4 in the direction of of $\mathbf{u}=\langle -1,1\rangle$.
Solution: $\langle -2\sqrt{2}, 2\sqrt{2}\rangle$

(c) Vector of length 2 in the direction of $\mathbf{v}=\mathbf{i}-\mathbf{j}$.
Solution: $\langle \sqrt{2},-\sqrt{2}\rangle$

(d) Unit vector in the direction opposite to $\mathbf{v}=\langle -2,4\rangle$.
Solution: $\langle \frac{1}{\sqrt{5}},-\frac{2}{\sqrt{5}}\rangle$.

(e) Vector $\mathbf{v}$ of length 2 making an angle of 30° with the x-axis.
Solution: $\langle \sqrt{3},1\rangle$.

(3) Determine whether or not the two vectors are parallel:

(a) $\mathbf{u}=\langle 1,-2,5\rangle$, $\mathbf{v}=\langle -2,4,-10\rangle$.
Solution: By definition the two vectors are parallel if there exists a scalar λ such that $u=\lambda v$. In other words, we need

$$\langle 1,-2,5\rangle = \langle -2\lambda, 4\lambda, -10\lambda\rangle$$

Two vectors are the same iff all their components are the same. Setting the first components equal we get $1=2\lambda \implies \lambda=-\frac{1}{2}$. If the vectors are indeed parallel, this is the only value of λ

that can work.

Plugging in this value in the vector equality above we see that it works, so the vectors are parallel.

(b) $\mathbf{u} = \langle 4, 2, -6\rangle$, $\mathbf{v} = \langle 2, 1, 3\rangle$.
SOLUTION: We proceed as above. The equation that we have to solve is

$$\langle 4, 2, -6\rangle = \langle 2\lambda, \lambda, 3\lambda\rangle$$

Again, we set the first components equal to get $4 = 2\lambda \Longrightarrow \lambda = 2$. Plugging $\lambda = 2$ in the vector equality above we get $\langle 4, 2, -6\rangle = \langle 4, 2, 6\rangle$. Notice that the third components do not match, so the vectors are not parallel.

SOME CALCULUS REVIEW

(1) Find the limits

- $\lim_{x\to\infty} \frac{x^4-3x}{5x^5-2} = \infty$
- $\lim_{x\to 0} \frac{3x^3-2}{4x^2+1} = -2$
- $\lim_{x\to 5} \frac{2x^2-24x+70}{x-5} = -4$
- $\lim_{x\to\infty} \frac{1}{x^2} ln(x) = 0$
- $\lim_{x\to 0} \frac{x}{sin(x)} = 1$

(2) What about some derivatives?

- $\frac{d}{dx} x^{\frac{3}{4}} e^x = \frac{3}{4} x^{\frac{-1}{4}} e^x + x^{\frac{3}{4}} e^x$
- $\frac{d}{dx} e^{-x^2} = -2xe^{-x^2}$
- $\frac{d}{dx} \frac{x}{cos(x)} = \frac{xsin(x)}{cos^2(x)} + \frac{1}{cos(x)}$
- $\frac{d}{dx} \sqrt{\frac{x-2}{x-3}} = \frac{1}{2} \sqrt{\frac{x-3}{x-2}} \frac{1}{(x-3)^2}$

(3) Compute the following integrals:

- $\int_0^5 \frac{x}{x^2+2} dx = \frac{1}{2} ln(27)$
- $\int_1^{e-1} \frac{x-1}{x^2-1} = 1 - ln(2)$
- $\int_0^{\sqrt{\pi}} xcos(x^2)dx = 0$
- $\int_2^5 (3x^2-2)e^{x-3}dx = 49e^2 - 4e^{-1}$

- $\int_0^{\pi} x sin(x) dx = \pi$
- $\int_1^{\infty} \frac{1}{x^2} ln(x) dx = 1$

(4) Two particles A and B move on a one dimensional string. The positions as functions of time are given by: $x_A = \frac{1}{\sqrt{2}}t$ and $x_B = sin(t)$. There is a time between 6 and 8 seconds when the distance of the particles is extremal (critical point of the distance function). Find it and determine if the distance is maximized or minimized at this given time.

SOLUTION: Between 6 and 8 seconds we have $\frac{1}{\sqrt{2}}t > sin(t)$. So the distance is given by $d(t) = \frac{1}{\sqrt{2}}t - sin(t)$. We find the critical points by setting the derivative equal to 0. Let us compute

$$\frac{d}{dt}d(t) = \frac{1}{\sqrt{2}} - cos(t) = 0$$

So we want to solve $cos(t) = \frac{1}{\sqrt{2}}$. The only t between 6 and 8 that satisfies this is $t = 2\pi + \frac{\pi}{4}$. In order to determine if it is a maximum or minimum, we use the second derivative test. We have $\frac{d^2}{dt^2}d(t) = sin(t)$. If we plug in $t = 2\pi + \frac{\pi}{4}$ we get $\frac{1}{\sqrt{2}}$, which is positive. This means that the function attains a minimum at $2\pi + \frac{\pi}{4}$.