Some flatness results

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Abstract

Here I record some results in commutative algebra/algebraic geometry that are related to the concept of flatness and that I have found useful at some point. This is my laundry list of criteria to try to prove that something is flat.

Proposition 1 (Local criterion for flatness). *See [Sta19, Tag 00ML].*

There are several variations of this. One common one goes as follows.

Proposition 2 (Infinitesimal criterion for flatness). *See [Sta19, Tag 0523].*

Another version of the local criterion is:

Proposition 3 (Slicing criterion for flatness). *See [Sta19, Tag 00ME]*

There are a few of results for checking flatness by passing to fibers.

Proposition 4 (Critere de platitude par fibres). *See [Sta19, Tag 039A].*

Proposition 5 (Flatness of relative complete intersections). *See [Sta19, Tag 00SW].*

There is the miracle flatness criterion, for which you only need to count dimensions.

Proposition 6 (Miracle flatness). *See [Sta19, Tag 00R4].*

There is a criterion for checking flatness over a base in terms of valuation rings, resembling the valuative criteria for properness/separatedness. The reference is [RG71, Corollaire 4.2.10]

Proposition 7 (Valuative criterion for flatness). *Let S be the spectrum of a local reduced ring. Let $p \in S$ denote the special point. Let X be a scheme of finite presentation over S. Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-sheaf of finite type. For any point $x \in X_p$, the following are equivalent*

(1) $\mathcal{F}$ is S-flat at x.

(2) *For any spectrum of a valuation ring S' with closed point p' and any local morphism of schemes $(S', p') \longrightarrow (S, p)$, the pullback $\mathcal{F}_{X_{S'}}$ is S'-flat at all points in the special fiber $X_{p'}$ that map to x.*

There is another flatness criterion in situations where we expect the morphism to be étale. Recall that a point y in a scheme Y is geometrically unibranch if the strict henselization of $\mathcal{O}_{Y,y}$ has a single minimal prime [Sta19, Tag 06DM] (if Y is reduced, this is the same as $(\mathcal{O}_{Y,y})^{sh}$ being an integral domain).

For the proof of the following, see [Gro67, Thm. 18.10.1]

Proposition 8. *Let $f : X \to Y$ be an unramified morphism of locally Noetherian schemes. Let x be point of X such that*

(1) *$\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is injective.*

(2) *$(\mathcal{O}_{Y,f(x)})^{sh}$ is an integral domain.*

Then, f is flat at x. In particular, a dominant unramified morphism $f : X \to Y$ between integral Noetherian schemes is automatically étale if Y is geometrically unibranch as in [Sta19, Tag 0BQ2].

If you know of other cool criteria for flatness, please let me know!

References

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