A note on scheme theoretic image

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Abstract

I am writing this short note to record some thoughts on scheme theoretic images. I worked out some of the following propositions when trying to define the derived subgroup of a group scheme of finite type over a field.

We start by introducing the main concept we will be discussing.

Definition 1. *Let $f : X \longrightarrow Y$ be a morphism of schemes. The scheme theoretic image is defined to be a closed subscheme $Im(f)$ of Y such that*

(i) f factors through the immersion $i : Im(f) \hookrightarrow Y$.

(ii) It is universal in the following sense. Suppose that we have a closed subscheme $j : M \hookrightarrow Y$ such that f factors though j. Then, in fact the closed immersion $i : Im(f) \hookrightarrow Y$ factors (uniquely) though j.

Lemma 2. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be maps of schemes. Let $i : Im(f) \hookrightarrow Y$ be the scheme theoretic image of f. Then, $Im(g \circ f) = Im(g \circ i)$.*

If $f : X \longrightarrow Y$ is a morphism of schemes and $p : \sqcup_i X_i \longrightarrow X$ is a Zariski covering, then the previous proposition implies that $\mathrm{Im}(f) = \mathrm{Im}(f \circ i)$. In particular, we can assume that the domain of f is a disjoint union of affine schemes.

It is well known that if f is quasicompact and quasiseparated, then taking schematic image commutes with flat base change. For general f this is far from true (in fact schematic image is not well bahaved when you pass to a Zariski cover, let alone a flat cover!). In order to prove some results for more general f, we need a concrete description of schematic image. But first we need some background.

Let X be an arbitrary scheme. The abelian category $\mathrm{QCoh}(X)$ of quasicoherent sheaves on X has a colimit preserving faithful embedding $\phi : \mathrm{QCoh}(X) \longrightarrow \mathcal{O}_X\text{-Mod}$ into the category of all $\mathcal{O}_X$-modules.

By a theorem of Gabber, $\mathrm{QCoh}(X)$ is a Grothendieck abelian category (the main result here is that there exists a generating set). We can therefore use the small object argument to conclude that there exists a functor $Q : \mathcal{O}_X\text{-Mod} \longrightarrow \mathrm{QCoh}(X)$ that is right adjoint to the embedding ϕ. This is called the coherator. Among other things, the existence of Q implies that $\mathrm{QCoh}(X)$ has all limits! For a discussion of the coherator, see [TT90, Appendix B] or [Sta18, Tag 08D6].

Proposition 3. *Let $f : X \longrightarrow Y$ a morphism of schemes. Then the quasicoherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_Y$ defining the scheme theoretic image $Im(f)$ is given by $Q(ker(\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X))$.*

In general, the coherator does not need to commute with passing to Zariski covers (same counterexamples as for schematic image). However, we can use the description of the coherator for quasicompact quasiseparated schemes in [TT90, Appendix B], plus flat base change to get the commutativity of the coherator in the special case when our sheaf is an intersection of quasicoherent sheaves and tensoring commutes with taking intersections. This is true for example when we base change by a ring that is free as a module over the base.

We can use this to show the following:

Proposition 4. *Let R be a ring and $f : X \longrightarrow Y$ be a morphism of R-schemes. Suppose that $Im(f)$ is quasicompact and quasiseparated. Let S be an R-algebra that is free as an R-module. Form the product map $f_S : X \times_R Spec\, S \longrightarrow Y \times_R Spec\, S$. Then, we have $Im(f_S) = Im(f) \times_R Spec\, S$.*

We can in particular use this to prove the following stronger result when the base ring R is actually a field k.

Proposition 5. *Let $f : X \longrightarrow Y$ be a morphism of k-schemes, where k is a field. Suppose that $Im(f)$ is quasicompact and quasiseparated. Let Z be an arbitrary k scheme. Form the product map $f \times id : X \times Z \longrightarrow Y \times Z$. Then, we have $Im(f \times Z) = Im(f) \times Z$.*

For the proof of this proposition, reduce to Z affine and X a union of affine schemes by using the universal property of schematic image and the Lemma above. (The same argument works for an arbitrary ring R if we know that there is a cover of Z by affine opens whose coordinate rings are free as R-modules.

Remark 6. *Notice that the hypothesis of the propositions are always satisfied if Y is quasicompact and quasiseparated.*

For example this tells us that if a (possibly infinite!) set of k-points in a quasicompact and quasi-separated k-scheme X is schematically dense, then it remains schematically dense after arbitrary base-change (I have heard someone call this "relatively schematically dense").

As a consequence of the proposition, we also get the following useful corollary.

Corollary 7. *Let $f_1 : X_1 \longrightarrow Y_1$ and $f_2 : X_2 \longrightarrow Y_2$ be maps of k-schemes. Suppose that $Im(f_1)$ and $Im(f_2)$ are quasicompact and quasiseparated. Then, $Im(f_1 \times f_2) = Im(f_1) \times Im(f_2)$.*

This corollary can be used, for example, to prove that the derived subgroup of a group scheme of finite type over k is indeed a group.

References

[Sta18] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2018.

[TT90] R. W. Thomason and Thomas Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pages 247–435. Birkhäuser Boston, Boston, MA, 1990.