Resolutions via monoids

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Abstract

Short note on canonical/ bar resolutions.

1 Monoidal categories and monoids

Definition 1.1. *Let $\mathcal{C}$ be a category. A strict monoidal structure on $\mathcal{C}$ consists of a functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ satisfying the following two conditions.*

(1) There is an object $1_{\mathcal{C}}$ in $\mathcal{C}$ such that we have equality of functors $1_{\mathcal{C}} \otimes (-) = (-) \otimes 1_{\mathcal{C}} = id_{\mathcal{C}}$.

(2) The following diagram commutes

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, "\otimes \times \mathrm{id}_{\mathcal{C}}"] \arrow[d, "\mathrm{id}_{\mathcal{C}} \times \otimes"] & \mathcal{C} \times \mathcal{C} \arrow[d, "\otimes"] \\ \mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"] & \mathcal{C} \end{tikzcd}$$

A category equipped with a strict monoidal structure will be called a strict monoidal category.

By the Yoneda lemma, the object $1_{\mathcal{C}}$ is unique up to isomorphism. We call it the unit of the monoidal structure. There is a natural dual notion of comonoid object obtained by inverting all arrows in the definition above.

Example 1.2. *Let $\mathcal{D}$ be any category. The category $End(\mathcal{D})$ of endofunctors has functors $F : \mathcal{D} \longrightarrow \mathcal{D}$ as objects. The morphisms are given by natural transformations. The category $End(\mathcal{D})$ is equipped with a strict monoidal structure $\circ$ given by compositions of functors.*

Definition 1.3. *Let $(\mathcal{C}, \otimes_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}})$ be two strict monoidal categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called monoidal if it preserves the monoidal structures. We let $Fun_{\otimes}(\mathcal{C}, \mathcal{D})$ be the full subcategory of $Fun(\mathcal{C}, \mathcal{D})$ consisting of monoidal functors.*

Let us fix for the remaining of this section a strict monoidal category $(C, \otimes)$.

Definition 1.4. *A monoid in $\mathcal{C}$ consits of the data of* *(i) An object M in $\mathcal{C}$.*

(ii) A morphism $e : 1_{\mathcal{C}} \to M$.

(iii) A morphism $m : M \otimes M \to M$.

The following two diagrams are required to commute.

$$\begin{tikzcd}
M \arrow[r, "e\otimes \mathrm{id}_M"] \arrow[d, "\mathrm{id}_M\otimes e"] \arrow[dr, "\mathrm{id}_M"] & M \otimes M \arrow[d, "m"] \\
M \otimes M \arrow[r, "m"] & M
\end{tikzcd}$$

$$\begin{tikzcd}
M \otimes M \otimes M \arrow[r, "m\otimes \mathrm{id}_M"] \arrow[d, "\mathrm{id}_M\otimes m"] & M \otimes M \arrow[d, "m"] \\
M \otimes M \arrow[r, "m"] & M
\end{tikzcd}$$

The monoid objects in $\mathcal{C}$ form a category $\mathrm{Monoid}_{\mathcal{C}}$. The morphisms consists of morphisms of the underlying objects that intertwine the monoid structures.

Definition 1.5. *Δ denotes the subcategory of Set given as follows. Its objects are finite ordered sets of the form $[n] = \{0, 1, ..., n\}$ for $n \geq 0$. The morphisms are order preserving maps between two such sets.*

Δ_{aug} denotes the category obtained from Δ by adjoining an additional object $[-1] := \emptyset$, which is initial in Δ_{aug}.

We recall that there are certain distinguished morphisms in Δ called the coface and codegeneracy maps. We will denote the coface maps by $d_i : [n] \longrightarrow [n+1]$ and the codegeneracy maps by $s_i : [n+1] \longrightarrow [n]$.

Definition 1.6. *A functor $F : \Delta_{aug} \longrightarrow \mathcal{C}$ is called a comsimplicial object in $\mathcal{C}$. We will usually write $F_n := F([n])$.*

There is a strict monoidal structure $+$ on Δ_{aug}. For any sets $[n], [m]$ in Δ_{aug}, we can set $[n] + [m] = [n + m + 1]$. One should think of $+$ as stacking the two sets one after the other, i.e. listing first the elements of $[n]$ and then listing the elements of $[m]$. For any two maps of sets $f : [n] \to [l]$ and $g : [m] \to [k]$, the corresponding morphism $f + g : [n] + [m] \to [l] + [k]$ is given by stacking the maps f, g one after the other. Notice that $[-1]$ is the unit of $+$.

Suppose that we are given a monoidal functor $F : \Delta_{aug} \to \mathcal{C}$. We must have $F_{-1} = 1_{\mathcal{C}}$. Let us set $M := F_0$. Notice that we must have $F{-1} = F([0] + [0]) = M \otimes M$. Note that there are unique maps of sets $[1] \to [0]$ and $[-1] \to [0]$. We can apply F in order to obtain morphisms $m : M \otimes M \to M$ and $e : 1_{\mathcal{C}} \to M$. The fact that F is a functor implies that (M, e, m) is a monoid in $\mathcal{C}$.

We can use this observation to obtain a functor $\mathfrak{U}_{\mathcal{C}} : \mathrm{Fun}_{\otimes}(\Delta_{aug}, \mathcal{C}) \longrightarrow \mathrm{Monoid}_{\mathcal{C}}$.

Proposition 1.7. *The functor $\mathfrak{U}_{\mathcal{C}}$ is an equivalence of categories.*

Proof. Let us describe a quaisinverse functor $\mathfrak{M}_{\mathcal{C}} : \mathrm{Monoid}_{\mathcal{C}} \longrightarrow \mathrm{Fun}_{\otimes}(\Delta_{aug}, \mathcal{C})$. Let M be a monoid in C. We define $\mathfrak{M}_{\mathcal{C}}(M)$ as follows. Set $\mathfrak{M}_{\mathcal{C}}(M)_n := M^{\otimes n+1}$. The coface maps $d_i : M^{\otimes n} \longrightarrow M^{\otimes n+1}$ are given by

$$d_i := \mathrm{id}_M^{\otimes i-1} \otimes e \otimes \mathrm{id}_M^{\otimes n-i}$$

The codegeneracies $s_i : M^{\otimes n+1} \longrightarrow M^{\otimes n}$ are given by

$$s_i := \mathrm{id}_M^{\otimes i-1} \otimes m \otimes \mathrm{id}_M^{\otimes n-i}$$

□

We can also use the category Δ_{aug}^{op} with its dual comonoid structure. This way we can obtain the analogue of Proposition 1.7 for comonoids.

Definition 1.8. *Let (M, e, m) be a monoid in $\mathcal{C}$. A M-algebra consists of the data of*

(i) An object A in $\mathcal{C}$.

(ii) A morphism $a : M \otimes A \to A$ (action morphism).

These are required to make the following two diagrams commute.

$$\begin{tikzcd}
A \arrow[r, "e\otimes\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A\otimes e"'] \arrow[dr, "\mathrm{id}_A"] & A \otimes M \arrow[d, "a"] \\
M \otimes A \arrow[r, "a"] & M
\end{tikzcd}$$

$$\begin{tikzcd}
M \otimes M \otimes A \arrow[r, "m\otimes\mathrm{id}_A"] \arrow[d, "\mathrm{id}_M\otimes a"'] & M \otimes A \arrow[d, "a"] \\
M \otimes A \arrow[r, "a"] & A
\end{tikzcd}$$

We have a dual notion of coalgebras over a comonoid.

For a given monoid (M, e, m), we will denote by $\mathcal{Alg}_M$ the category of M-algebras. Morphisms are required to intertwine the action morphisms. There is an obvious functor $U_M : \mathcal{Alg}_M \longrightarrow \mathcal{C}$ given by forgetting the algebra structure.

Proposition 1.9. *The functor U_M admits a left adjoint $F_M : \mathcal{C} \longrightarrow \mathcal{Alg}_M$.*

Proof. We describe F_M, as well as the unit and counit morphisms. Let C be an object in $\mathcal{C}$. The algebra $F_M(C)$ will have underlying object $M \otimes C$ in $\mathcal{C}$. The action map is given by $a : M \otimes M \otimes C \xrightarrow{m\otimes\mathrm{id}_C} M \otimes C$.

The unit $\xi : \mathrm{Id}_{\mathcal{C}} \Rightarrow U_M F_M$ is given as follows. For any algebra $C \in \mathrm{Ob}(\mathcal{C})$, we set

$$\eta(A) : C \xrightarrow{e \otimes \mathrm{id}_C} M \otimes C$$

The counit $\eta : F_M U_M \Rightarrow \mathrm{Id}_{\mathcal{Alg}_M}$ is given as follows. Let A be a M-algebra with action map $a : M \otimes A \longrightarrow A$. We define

$$\eta(A) : M \otimes A \xrightarrow{a} A$$

$\square$

2 Resolutions

Let us fix a category $\mathcal{C}$.

Definition 2.1. *A (co)monad on $\mathcal{C}$ is a (co)monoid object in the strict monoidal category $(End(\mathcal{C}), \circ)$. In detail, a monad consists of the data of*

(i) An endofunctor $T : \mathcal{C} \to \mathcal{C}$.

(ii) A natural transformation $m : T \circ T \Rightarrow T$ (the multiplication).

(iii) A natural transformation $e : id_T \Rightarrow T$ (the identity).

They satisfy conditions as in Definition 1.4.

Example 2.2. *Adjunctions between functors provide a natural source of (co)monads. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be a pair of adjoint functors. Recall that the unit of the adjunction is a natural transformation $\xi : id_{\mathcal{C}} \Rightarrow UF$. The counit is a natural transformation $\eta : FU \Rightarrow id_{\mathcal{D}}$. We can compose with F and U to obtain natural transformations*

$$F\xi U : FU \Rightarrow (FU) \circ (FU)$$

$$U\eta F : (UF) \circ (UF) \Rightarrow UF.$$

The properties of the unit and counit imply that $(UF, \xi, U\eta F)$ is a monad on $\mathcal{D}$. Similarly $(FU, \eta, F\xi U)$ is a comonad in $\mathcal{C}$.

Definition 2.3. *Let T be a monad on $\mathcal{C}$. An algebra over T consits of the data of*

(i) An object A in $\mathcal{C}$.

(ii) A morphism $a : TA \to A$ (action morphism).

These are required to make the following two diagrams commute.

$$\begin{tikzcd} A \arrow[r, "e(A)"] \arrow[dr, "\mathrm{id}_A"'] & TA \arrow[d, "a"] \\ & A \end{tikzcd}$$

$$\begin{array}{ccc} T^2 \otimes A & \xrightarrow{m(A)} & TA \\ \downarrow{\scriptstyle T(a)} & & \downarrow{\scriptstyle a} \\ TA & \xrightarrow{\quad a \quad} & A \end{array}$$

The collection of algebras over T form a category $\mathcal{Alg}_T$. There is a forgetful map $U_F : \mathcal{Alg}_T \longrightarrow \mathcal{C}$. The same argument as in Proposition 1.9 yields the following.

Proposition 2.4. *The forgetful functor U_T admits a left adjoint $F_T : \mathcal{C} \longrightarrow \mathcal{Alg}_T$.*

Let T be a monad on $\mathcal{C}$. Proposition 1.7 yields an augmented cosimplicial object $\mathfrak{M}_{\mathrm{End}(\mathcal{C})}(T) : \Delta_{aug} \longrightarrow \mathrm{End}(\mathcal{C})$. Recall that the image of the object $[-1]$ is $\mathrm{id}_{\mathcal{C}}$. We can use currying to interpret this as a functor $\mathfrak{M}_{\mathrm{End}(\mathcal{C})}(T) : \mathcal{C} \longrightarrow \mathrm{Fun}(\Delta_{aug}, \mathcal{C})$. In plain words, this yields a functorial assignment of an augmented cosimplicial object for all $C \in \mathcal{C}$. We call $\mathfrak{M}_{\mathrm{End}(\mathcal{C})}(T)\,(C)$ the canonical cosimplicial resolution associated to T. We denote it by $\mathrm{Can}_T(C)$. Notice that depends functorially on the monad. If we have a morphism of monads $T \to S$ (i.e. a morphism of monoid objects in $\mathrm{End}(\mathcal{C})$) we get an associated map of canonical resolutions $\mathrm{Can}_T(-) \Rightarrow \mathrm{Can}_S(-)$.

This construction is usually applied to (co)monads coming from adjunctions as in Example 2.2. Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D} :U$ be a pair of adjoint functors. Recall that this yields a monad UF on $\mathcal{C}$. For each $C \in \mathrm{Ob}\,\mathcal{C}$, we get a canonical cosimplicial resolution $\mathrm{Can}_{UF}(C)$. Let us give a detailed description using the construction of $\mathfrak{M}_{\mathrm{End}(\mathcal{C})}$ in Proposition 1.7.

We have $\mathrm{Can}_{UF}(C)_n = (UF)^{\circ(n+1)}(C)$. The coface map $d_i : (UF)^{\circ n}(C) \longrightarrow (UF)^{\circ n+1}(C)$ is

$$d_i = (UF)^{\circ i-1}\,\xi\,(UF)^{\circ n-i}(C)$$

The codegeneracy maps $s_i : (UF)^{\circ n+1}(C) \longrightarrow (UF)^{\circ n}(C)$ are given by

$$s_i = (UF)^{\circ i-1}\,U\,\eta\,F\,(UF)^{\circ n-i}(C)$$

For all $n \geq 0$, we have that $\mathrm{Can}_{UF}(C)_n$ is in the image of U. It is sometimes the case that the essential image of U consists of objects that are well-behaved (e.g. injective sheaves, acyclic objects with respect to some functor ...). In this case $\mathrm{Can}_{UF}(C)$ provides us with a resolution in terms of such well-behaved objects. The next step is to show"exactness" of such resolutions. For this we first need some terminology.

Definition 2.5. *The category Δ_{aug}^{spl} has the same objects as Δ_{aug}. The morphisms are consits of the morphisms of Δ_{aug} plus "extra" codegeneracy morphism $s_{-1} : [n] \longrightarrow [n-1]$ for all $n \geq 0$. This morphisms satisfy the following identities*

(a) $s_{-1}d_0 = id_{[n-1]}$ *for all* $n \geq 0$.

(b) $s_{-1}d_{i+1} = d_i s_{-1}$ *for all* i.

(c) $s_{-1}s_{i+1} = s_i s_{-1}$ *for all* i.

We say that an augmented cosimplicial object splits if it extends to a covariant functor from the larger category Δ_{aug}^{spl}.

Remark 2.6. *See [GJ09] page 199 or [Rie14]. Apparently there are several nonequivalent definitions of contractible simplicial set, as show in [MB].*

This is a natural notion of exactness for (co)simplicial objects, as the next couple of examples show.

Example 2.7 (Acyclicity of Moore complex)**.** *Let $\mathcal{C}$ be an abelian category. Let $C_\bullet : \Delta_{aug} \longrightarrow \mathcal{C}$ be a cosimplicial object. We can form the corresponding Moore complex as in [GJ09] pg. 146. The differential is given by the alternating sum of the face maps. If $C_\bullet$ is split, then the Moore complex admits a chain homotopy [GJ09] Lemma 2.15 in page 160. In particular it is acyclic.*

Example 2.8 (Descent for morphisms that admit a section)**.** *Let S be a scheme. Set $\mathcal{C} := Sch/S$ to be the category of schemes over S. Let $\mathcal{X}$ be a contravariant groupoid-valued pseudofunctor on Sch/S (e.g. an algebraic stack over S). Let $C_\bullet : \Delta_{aug} \longrightarrow Sch/S$ be an augmented simplicial object. If $C_\bullet$ admits a splitting, then there is an equivalence of groupoids $Fun(C_\bullet, \mathcal{X}) \simeq \mathcal{X}(C_{-1})$.*

We end this section by showing that canonical resolutions associated to adjoint pairs are not very far from being split.

Proposition 2.9. *Let $F : \mathcal{C} \underset{\perp}{\rightleftarrows} \mathcal{D} : U$ be a pair of adjoint functors. Fix an object C in $\mathcal{C}$. Consider the canonical resolution $Can_{UF}(C)$. The comsimplicial object in $\mathcal{D}$ given by the composition composition $F \circ Can_{UF}(C)$ admits a splitting.*

If the functor F is"faithfully-exact" in some sense, then we can conclude exactness of the canonical resolution. We refer to the next section for examples clarifying how this works in practice.

For completeness, we state the analogous result for the comonad FU. Let D be an object in $\mathcal{D}$. We denote by $\mathrm{Can}_{FU}(D) : \Delta_{aug}^{op} \longrightarrow \mathcal{D}$ the corresponding augmented simplicial object in $\mathcal{D}$.

Proposition 2.10. *Let $F : \mathcal{C} \underset{\perp}{\rightleftarrows} \mathcal{D} : U$ be a pair of adjoint functors. Fix an object D in $\mathcal{D}$. Consider the canonical resolution $Can_{FU}(D)$. The agumented simplicial object in $\mathcal{C}$ given by the composition composition $U \circ Can_{FU}(D)$ admits a splitting.*

Remark 2.11. *The canonical resolution is universal among resolutions satisfying an 'acyclicity" condition closely related to the splitting in Propositions 2.9 and 2.10. For more details, see [nLa].*

3 Examples

In this section we give several applications of the ideas we have discussed. We show how one can exploit naturally arising monoid objects in order to produce functorial resolutions in algebra and algebraic geometry.

In practice one often encounters (nonstrict) monodoidal categories. This means that there are canonical associativity and unit isomorphisms that satisfy certain coherence conditions. All of our discussion for strict categories applies verbatim to nonstrict categories. One only needs to compose with the appropriate isomorphisms when forming the corresponding resolutions.

Example 3.1 (Cech nerve). *One can easily obtain monoidal structures in categories with fiber products. Let us give an example. Let S be a scheme, and set $\mathcal{C} = Sch/S$ to be the category of schemes over S. The fiber product $\times_S$ gives Sch_S the structure of a monoidal category. Any object $f : X \to S$ in Sch/S admits the structure of a comonoid with multiplication map given by the diagonal $\Delta : X \longrightarrow X \times_S X$. The dual of Proposition 1.7 yields an augmented simplicial scheme*

$$... \; X \times_S X \times_S X \substack{\rightarrow\\ \rightarrow\\ \rightarrow} X \times_S X \rightrightarrows X \; \longrightarrow \; S$$

This is usually called the Cech nerve of the map $f : X \longrightarrow S$. Any morphism $X \longrightarrow Y$ in Sch_S preserves the comonoid structures. Therefore the formation of the Cech nerve is functorial in Sch/S.

For the rest of our examples, we look for adjoint pairs of functors in order to produce canonical resolutions.

Example 3.2 (Godement resolution). *Let X be a topological space. Let us denote by X^{disc} the discrete topological space with the same underlying set as X. There is a canonical continuous map $f : X^{disc} \longrightarrow X$. Let $Sh(X)$ (resp. $Sh(X^{disc})$) denote the category of sheaves of abelian groups on X (resp. X^{disc}). The continuous map f induces a pair of additive functors $f^{-1} : Sh(X) \longrightarrow Sh(X^{disc})$ and $f_* : Sh(X^{disc}) \longrightarrow Sh(X)$. For any sheaf $\mathcal{F} \in Sh(X)$, we obtain a canonical cosimplicial resolution $Can_{f_*f^{-1}}(\mathcal{F})$. We will let $MCan_{f_*f^{-1}}(\mathcal{F})$ denote the associated Moore complex, as in Example 2.7.*

We claim that $MCan_{f_*f^{-1}}(\mathcal{F})$ is exact. Notice that it suffices to check that $f^{-1}MCan_{f_*f^{-1}}(\mathcal{F})$ is acyclic, because exactness can be checked at the level of stalks. But $f^{-1}MCan_{f_*f^{-1}}(\mathcal{F}) = M\, f^{-1} \circ Can_{f_*f^{-1}}(\mathcal{F})$. By Proposition 2.9, the cosimplicial set $f^{-1} \circ Can_{f_*f^{-1}}(\mathcal{F})$ is split. Therefore its Moore complex $M\, f^{-1} \circ Can_{f_*f^{-1}}(\mathcal{F})$ is acyclic by Example 2.7. Hence $MCan_{f_*f^{-1}}(\mathcal{F})$ is a resolution of $\mathcal{F}$.

$$MCan_{f_*f^{-1}}(\mathcal{F}) : \quad 0 \to \mathcal{F} \to f_*f^{-1}\mathcal{F} \to (f_*f^{-1})^2\mathcal{F} \to ...$$

All sheaves on X^{disc} are flasque. Since f_* preserves flasque sheaves, we see that the image of f_* consists of flasque sheaves. In particular $MCan_{f_*f^{-1}}(\mathcal{F})$ is a functorial resolution of $\mathcal{F}$ by flasque sheaves.

Another example is the concept of canonical free resolutions. Let $\mathcal{D}$ be a category of algebraic objects. It is usually the case that there is a natural forgetful functor $U : \mathcal{D} \to \mathcal{C}$ into a category of algebraic objects with less structure. In such situations, there is often a left adjoint functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ given by some kind of free object construction. Let D be an object of $\mathcal{D}$. We get a functorial resolution $\mathrm{Can}_{FU}(D)$ by free objects

$$\mathrm{Can}_{FU}(D): \ \ ... \, (FU)^3(D) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} (FU)^2(D) \rightrightarrows (FU)(D) \longrightarrow D$$

Proposition 1.9 provides a good source of fogertful/free adjunctions. Let $(\mathcal{C}, \otimes)$ be a strict monoidal category. Fix a monoid (M, e, m) in $\mathcal{C}$. Recall that there is an adjunction $F_M \colon \mathcal{C} \rightleftarrows \mathcal{A}lg_M : U_M$. For any M-algebra A, we obtain a canonical simplicial resolution by free M-algebras

$$\mathrm{Can}_{F_M U_M}(A): \ \ ... \, (M)^{\otimes 3} \otimes A \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} (M)^{\otimes 2} \otimes A \rightrightarrows M \otimes A \longrightarrow A$$

Example 3.3 (Bar Complex)**.** *Let k be a commutative ring. Set $\mathcal{C} = k - Mod$ to be the category of k-modules. It is a (nonstrict) monoidal category via the tensor product $\otimes_k$. A monoid M in $(k - Mod, \otimes_k)$ is the same as an associative unital algebra over k. The category of M-algebras $\mathcal{A}lg_M$ is the category of left modules over M. Let V be a left M-module. By taking the Moore complex of the corresponding agumented simplicial set, we obtain a free resolution of M-modules*

$$...M^{\otimes 3} \otimes_k V \longrightarrow M^{\otimes 2} \otimes_k V \longrightarrow M \otimes_k V \longrightarrow V$$

Proposition 2.10 implies that this is an exact complex of k-modules. This of course also implies that it is an exact complex of M-modules (the forget functor U_M is "faithfully-exact").

Let G be a group. As a special case of Example 3.3 we can let k be a field and $M = k[G]$ be the group algebra of G. We then recover the canonical resolution that computes group cohomology.

References

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