HOMEWORK SOLUTIONS — MATH 1910

Sections 5.1, 5.2 — Fall 2016

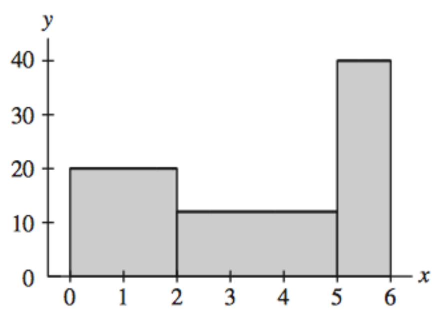
Problem 5.1.2

An ostrich runs with velocity 20 km/h for 2 minutes (min), 12 km/h for 3 min, and 40km/h for another minute. Compute the total distance traveled and indicate with a graph how this quantity can be interpreted as an area.

SOLUTION. The total distance traveled by the ostrich is

$$\left(\frac{20}{60}\right)(2) + \left(\frac{12}{60}\right)(3) + \left(\frac{40}{60}\right)(1) = \frac{2}{3} + \frac{3}{5} + \frac{2}{3} = \frac{29}{15}\text{km}$$

This distance is the area under the graph below which shows the ostrich's velocity as a function of time.



5.1.2

Problem 5.1.5

Compute R_5 *and* L_5 *over* $[0, 1]$ *using the following values:*

x	0	0.2	0.4	0.6	0.8	1
$f(x)$	50	48	46	44	42	40

SOLUTION. We have $\Delta x = \frac{1-0}{5} = 0.2$. Hence we have

$$L_5 = 0.2(50 + 48 + 46 + 44 + 42) = 0.2(230) = 46$$

and

$$R_5 = 0.2(48 + 46 + 44 + 42 + 40) = 0.2(220) = 44.$$

5.1.5

Problem 5.1.42

Use linearity and formulas (3)-(5) to rewrite and evaluate the sum

$$\sum_{m=1}^{20}\left(5+\frac{3m}{2}\right)^2$$

SOLUTION.

$$\begin{aligned}\sum_{m=1}^{20}\left(5+\frac{3m}{2}\right)^2 &= \sum_{m=1}^{20}\left(25+15m+\frac{9}{4}m^2\right)\\ &= 25\sum_{m=1}^{20}1+15\sum_{m=1}^{20}m+\frac{9}{4}\sum_{m=1}^{20}m^2\\ &= 25(20+15\left(\frac{20^2}{2}+\frac{20}{2}\right)+\frac{9}{4}\left(\frac{20^3}{3}+\frac{20^2}{2}+\frac{20}{6}\right)\\ &= 10107.5 \qquad \boxed{5.1.42}\end{aligned}$$

Problem 5.1.86

Prove that for any function f *on* $[a, b]$,

$$R_N - L_N = \frac{b-a}{N}\left(f(b)-f(a)\right).$$

SOLUTION. Let $\Delta x = (b-a)/N$ and partition the interval $[a, b]$ into N equal-length subintervals with endpoints

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b.$$

We have $L_N = \Delta x \sum_{k=0}^{N-1} f(x_k)$ and $R_N = \Delta x \sum_{k=1}^{N} f(x_k)$, so

$$\begin{aligned}R_N - L_N &= \Delta x\left(\sum_{k=1}^{N} f(x_k) - \sum_{k=0}^{N-1} f(x_k)\right)\\ &= \Delta x\left(f(x_N)+\sum_{k=1}^{N} f(x_k) - \left(f(x_0)+\sum_{k=0}^{N-1} f(x_k)\right)\right)\\ &= \Delta x\left(f(x_N)-f(x_0)\right)\\ &= \frac{b-a}{N}\left(f(b)-f(a)\right). \qquad \boxed{5.1.86}\end{aligned}$$

Problem 5.1.88

Use [the previous problem] to show that if f *is positive and monotonic, then the area* A *under its graph over* $[a, b]$ *satisfies*

$$|R_N - A| \le \frac{b-a}{N}|f(b) - f(a)|.$$

SOLUTION. Suppose f is positive and increasing. Then R_N is an over-approximation of the area under the graph of f, and L_N is an under-approximation. That is, we have

$$L_N \le A \le R_N.$$

Subtracting R_N from all three parts of that inequality yields

$$L_N - R_N \le A - R_N \le 0,$$

and negating all three parts gives

$$0 \le R_N - A \le R_N - L_N.$$

From the previous problem we have

$$0 \le R_N - A \le \frac{b-a}{N}\left(f(b) - f(a)\right),$$

and taking absolute values gives the desired result.

$$|R_N - A| \le \frac{b-a}{N}|\left(f(b) - f(a)\right)|$$

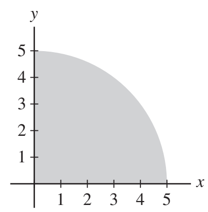
(Since $R_N - A$ and $f(b) - f(a)$ are positive, they are equal to their own absolute values.) The case in which f is decreasing is similar. 5.1.88

Problem 5.2.7

Draw a graph of the signed area represented by the integral and compute it using geometry:

$$\int_0^5 \sqrt{25 - x^2}\, dx.$$

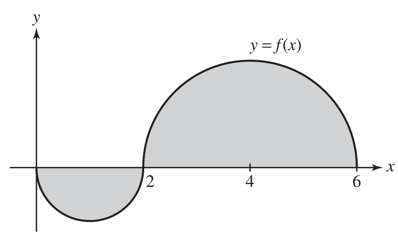
SOLUTION. The region below $y = \sqrt{25 - x^2}$ over $[0, 5]$ is a quarter of the circle of radius 5 centered at the origin:



Hence $\int_0^5 \sqrt{25 - x^2}\, dx = \frac{1}{4}\pi(5)^2 = \frac{25\pi}{4}$. $\boxed{5.2.7}$

Problem 5.2.14

Let $f(x)$ *figure shown in the figure below.*



Compute ***(a)*** $\int_1^4 f(x)\, dx$ ***(b)*** $\int_1^6 |f(x)|\, dx$

SOLUTION. **(a)** The region between the graph of f and x-axis consists of a quarter of a circle of radius 1 and a quarter of a circle of radius 2. Remembering that the definite integral computes *signed* area, we have

$$\int_1^4 f(x)\, dx = \frac{1}{4}\pi(2)^2 - \frac{1}{4}\pi(1)^2 = \frac{3}{4}\pi.$$

(b) For this integral, we compute the *unsigned* area between the graph of f and the x-axis, which consists of a quarter circle or radius 1 and a semicircles of radius 2.

$$\int_1^6 f(x)\, dx = \frac{1}{4}\pi(1)^2 + \frac{1}{2}\pi(2)^2 = \frac{9}{4}\pi. \qquad \boxed{5.2.14}$$

Problem 5.2.30

Determine the sign of the integral without calculating it. Draw a graph if necessary. $\int_{-2}^{2} x^3\,dx$

SOLUTION. By symmetry, the positive area from the interval $[0, 1]$ is cancelled by the negative area from $[-1, 0]$. With the interval $[-2, -1]$ contributing more negative area, the definite integral must be negative. 5.2.30

Problem 5.2.74

Calculate the integral: $\int_{0}^{2} |x^2 - 1|\,dx$.

SOLUTION. Since $x^2 - 1$ is negative on $[0, 1]$ and positive on $[1, 2]$, we have

$$|x^2 - 1| = \begin{cases} -(x^2 - 1) & 0 \le x \le 1, \\ x^2 - 1 & 1 \le x2. \end{cases}$$

Hence

$$\begin{aligned} \int_0^2 |x^2 - 1|\,dx &= \int_0^1 (1 - x^2)\,dx + \int_1^2 (x^2 - 1)\,dx \\ &= \left[x - \frac{1}{3}x^3\right]_{x=0}^{1} + \left[\frac{1}{3}x^3 - x\right]_{x=1}^{2} \\ &= 2. \end{aligned}$$

5.2.74

Problem 5.2.78

Prove that $0.277 \le \int_{\pi/8}^{\pi/4} \cos x\,dx \le 0.363.$

SOLUTION. $\cos x$ is decreasing on the interval $[\pi/8, \pi/4]$. Hence, for $\pi/8 \le x \le \pi/4$,

$$\cos(\pi/4) \le \cos x \le \cos(\pi/8).$$

Since $\cos(\pi/4) = \sqrt{2}/2$,

$$0.277 \le \frac{\pi}{8} \cdot \frac{\sqrt{2}}{2} = \int_{\pi/8}^{\pi/4} \frac{\sqrt{2}}{2}\,dx \le \int_{\pi/8}^{\pi/4} \cos x\,dx.$$

Since $\cos(\pi/8) \le 0.924$,

$$\int_{\pi/8}^{\pi/4} \cos x\,dx \le \int_{\pi/8}^{\pi/4} 0.924\,dx = \frac{\pi}{8}(0.924) \le 0.363.$$

Therefore, $0.277 \le \int_{\pi/8}^{\pi_4} \cos x \le 0.363.$ 5.2.78

Problem 5.2.82 *State whether true or false. If false, sketch the graph of a counterexample.*

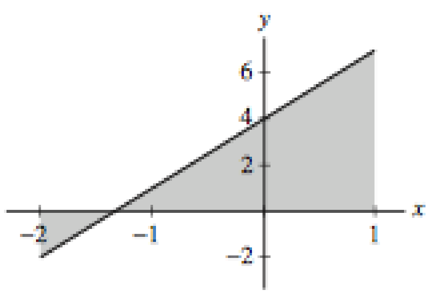
(a) If $f(x) > 0$*, then* $\int_a^b f(x)\,dx > 0$*.*

(b) If $\int_a^b f(x)\,dx > 0$*, then* $f(x) > 0$*.*

SOLUTION.

(a) This is **true** *in the case that* $b > a$. If $a > b$, then $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ and the integral is negative.

(b) It is **false** that if $\int_a^b f(x)\,dx > 0$, then $f(x) > 0$ for $x \in [a, b]$. A counterexample is $f(x) = 3x+4$ with $a = -2$ and $b = 1$. We see that $\int_{-2}^{1}(3x+4)dx = 7.5 > 0$, yet $f(-2) = -2 < 0$. Here is the graph.



5.2.82