Problem 8.7.4

Determine whether $\int_0^3 \frac{dx}{(3-x)^{3/2}}$ *converges by computing*

$$\lim_{R\to 3^-} \int_0^R \frac{dx}{(3-x)^{3/2}}$$

SOLUTION. Choose $u = 3 - x$, then $du = -\,dx$. We get

$$\int \frac{dx}{(3-x)^{3/2}} = -\int \frac{du}{u^{3/2}} = \frac{2}{\sqrt{u}} + C = \frac{2}{\sqrt{3-x}} + C$$

Thus,

$$\lim_{R\to 3^-} \int_0^R \frac{dx}{(3-x)^{3/2}} = \lim_{R\to 3^-} \frac{2}{\sqrt{3-x}}\Big|_0^R = -\frac{2}{\sqrt{3}} + \lim_{R\to 3^-} \frac{2}{\sqrt{3-R}} = \infty$$

Thus, the integral diverges. 8.7.4

Problem 8.7.28

Determine whether the improper integral converges and, if so, evaluate it:

$$\int_3^6 \frac{x\,dx}{\sqrt{x-3}}$$

SOLUTION. Note that $\lim_{x\to 3^+} \frac{x}{\sqrt{x-3}} = \infty$, so

$$\int_3^6 \frac{x\,dx}{\sqrt{x-3}} = \lim_{R\to 3^+} \int_R^6 \frac{x\,dx}{\sqrt{x-3}}$$

Choose $u = x - 3$, so $du = dx$, and

$$\int \frac{x\,dx}{\sqrt{x-3}} = \int \frac{u+3}{\sqrt{u}}\,du = \int (u^{1/2}+3u^{-1/2})\,du = \frac{2}{3}u^{3/2}+6u^{1/2}+C = \frac{2}{3}(x-3)^{3/2}+6(x-3)^{1/2}+C$$

So,

$$\int_3^6 \frac{x\,dx}{\sqrt{x-3}} = \lim_{R\to 3^+} \int_R^6 \frac{x\,dx}{\sqrt{x-3}} = \lim_{R\to 3^+} (\frac{2}{3}(x-3)^{3/2} + 6(x-3)^{1/2})\Big|_R^6$$

$$= \lim_{R\to 3^+} (\frac{2}{3}(3)^{3/2} + 6(3)^{1/2} - \frac{2}{3}(R-3)^{3/2} - 6(R-3)^{1/2}) = 2(3)^{1/2} + 6(3)^{1/2} + 0 + 0 = 8\sqrt{3}$$

So the integral converges to $8\sqrt{3}$. 8.7.28

Problem 8.7.57

Show that $0 \leq e^{-x^2} \leq e^{-x}$ *for* $x \geq 1$. *Use the Comparison Test to show that* $\int_0^\infty e^{-x^2}\,dx$ *converges.*

SOLUTION. For $x \geq 1$, $x^2 \geq x$. Thus, since $e \geq 1$, we have $0 \leq e^{-x^2} \leq e^{-x}$ for $x \geq 1$. Then, we find that

$$\int_1^\infty e^{-x}\,dx = \lim_{R\to\infty}\int_1^R e^{-x}\,dx = \lim_{R\to\infty} -e^{-x}\Big|_1^R = e^{-1} - \lim_{R\to\infty} -e^{-R} = e^{-1}$$

So $\int_1^\infty e^{-x}\,dx$ converges. So, by the comparison test, $\int_1^\infty e^{-x^2}\,dx$ converges.

Now, the function $f(x) = e^{-x^2}$ is continuous on $[0,1]$, so it is integrable on $[0,1]$, so $\int_0^1 e^{-x^2}\,dx$ is finite. Thus, rewriting

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

we find that the right hand side is a sum of two finite terms, so the whole thing is finite. Thus $\int_0^\infty e^{-x^2}\,dx$ converges. 8.7.57

Problem 8.7.82 *Find the volume of the solid obtained by rotating the region below the graph of* $y = e^{-x}$ *about the* x*-axis for* $0 \leq x < \infty$.

SOLUTION. Using the disk method, the volume is given by

$$V = \int_0^\infty \pi(e^{-x})^2\,dx = \pi\int_0^\infty e^{-2x}\,dx$$

First, compute the volume over a finite interval:

$$\pi\int_0^R e^{-2x}\,dx = \frac{-\pi}{2}e^{-2x}\Big|_0^R = \frac{-\pi}{2}(e^{-2R}-1) = \frac{\pi}{2}(1-e^{-2R})$$

Thus,

$$V = \lim_{R\to\infty}\pi\int_0^R e^{-2x}\,dx = \lim_{R\to\infty}\frac{\pi}{2}(1-e^{-2R}) = \frac{\pi}{2}(1-0) = \frac{\pi}{2}$$

8.7.82

Problem 8.7.84

The solid S obtained by rotating the region below the graph of $y = x^{-1}$ *about the* x*-axis for* $1 \leq x < \infty$ *is called* ***Gabriel's Horn***.

(a) *Use the Disk Method to compute the volume of S. Note that the volume is finite even though S is an infinite region.*

(b) *It can be shown that the surface area of S is*

$$A = 2\pi\int_1^\infty x^{-1}\sqrt{1+x^{-4}}\,dx$$

Show that A is infinite.

SOLUTION. (a) The height $h(x)$ of a disk, for $1 \le x < \infty$, is given by $h(x) = x^{-1}$, so the volume of the region is

$$V = \pi \int_1^{\infty} x^{-2}\, dx = \pi \lim_{R\to\infty} \int_1^R x^{-2}\, dx = -\pi \lim_{R\to\infty} x^{-1}\Big|_1^R$$
$$= -\pi \lim_{R\to\infty} (R^{-1} - 1) = \pi$$

(b) For $x > 1$, we have

$$x^{-1}\sqrt{1+x^{-4}} = \frac{\sqrt{1+\frac{1}{x^4}}}{x} = \frac{\sqrt{\frac{x^4+1}{x^4}}}{x}$$
$$= \frac{\sqrt{1+x^4}}{x^3} \ge \frac{\sqrt{x^4}}{x^3} = \frac{x^2}{x^3} = x^{-1}.$$

Since $\int_1^{\infty} x^{-1}\, dx$ diverges, so must A, by the Comparison Test.

8.7.84

Problem 8.7.93

Let $J_n = \int_0^{\infty} x^n e^{-x\alpha}\, dx$, where $n \ge 1$ is an integer and $\alpha > 0$. Prove that

$$J_n = \frac{n}{\alpha} J_{n-1}$$

and $J_0 = 1/\alpha$. Use this to compute J_4. Show that $J_n = n!/\alpha^{n+1}$.

SOLUTION. Using integration by parts, with $u = x^n$, $dv = e^{-x\alpha}dx$, we have $du = nx^{n-1}dx$ and $v = -\frac{1}{\alpha}e^{-x\alpha}$, so

$$\int x^n e^{-x\alpha}\, dx = -\frac{1}{\alpha} x^n e^{-x\alpha} + n/\alpha \int x^{n-1} e^{-x\alpha}\, dx$$

Thus,

$$J_n = \int_0^{\infty} x^n e^{-x\alpha}\, dx = \lim_{R\to\infty} -\frac{1}{\alpha} x^n e^{-x\alpha}\Big|_0^R + n/\alpha \int_0^{\infty} x^{n-1} e^{-x\alpha}\, dx$$
$$= \lim_{R\to\infty} \frac{-1}{\alpha} R^n e^{-R\alpha} + 0 + \frac{n}{\alpha} J_{n-1}$$

To evaluate the limit, rewrite $\lim_{R\to\infty} \frac{-1}{\alpha} R^n e^{-R\alpha} = \lim_{R\to\infty} \frac{-R^n}{\alpha e^{R\alpha}}$, which is of the indeterminate form $-\infty/\infty$, so we can apply L'Hopital's rule. In fact, we will have to apply L'Hopital's rule repeatedly:

$$\lim_{R\to\infty} \frac{-R^n}{\alpha e^{R\alpha}} = \lim_{R\to\infty} \frac{-nR^{n-1}}{\alpha^2 e^{R\alpha}} = \lim_{R\to\infty} \frac{-n(n-1)R^{n-2}}{\alpha^3 e^{R\alpha}} = ...$$
$$= \lim_{R\to\infty} \frac{-n(n-1)(n-2)...(2)(1)}{\alpha^{n+1} e^{R\alpha}} = 0$$

So $J_n = \frac{n}{\alpha} J_{n-1}$.

Now,

$$J_0 = \int_0^\infty e^{-x\alpha}\,dx = \lim_{R\to\infty}\int_0^R e^{-x\alpha}\,dx = \lim_{R\to\infty}\frac{-1}{\alpha}e^{-x\alpha}\Big|_0^R$$
$$= \lim_{R\to\infty}\frac{-e^{-R\alpha}}{\alpha} + \frac{1}{\alpha} = 0 + \frac{1}{\alpha} = \frac{1}{\alpha}$$

So

$$J_1 = \frac{1}{\alpha}J_0 = 1/\alpha^2$$

,

$$J_2 = \frac{2}{\alpha}J_1 = 2/\alpha^3$$

,

$$J_3 = \frac{3}{\alpha}J_2 = 6/\alpha^4$$

$$J_4 = \frac{4}{\alpha}J_3 = 24/\alpha^5$$

Now, we can prove by induction that $J_n = n!/\alpha^{n+1}$. First note that for $n = 0$, $J_n = J_0 = 1/\alpha = 0!/\alpha^{0+1}$, as needed. (It is a convention that $0! = 1$.)

Now, suppose, for $n > 1$, that $J_{n-1} = (n-1)!/\alpha^n$. Then

$$J_n = \frac{n}{\alpha}J_{n-1} = n((n-1)!)/\alpha^{n+1} = n!/\alpha^{n+1}$$

8.7.93

Problem 8.8.6

Find a constant C such that p is a probability density function on the given interval, and compute the probability indicated:

$p(x) = Ce^{-x}e^{-e^{-x}}$ on $(-\infty,\infty)$; $P(-4 \le X \le 4)$

SOLUTION. For any value of C, $p(x)$ will be continuous; and as long as $C \ge 0$, $p(x) \ge 0$ for all x. So we need to find a nonnegative C such that $\int_{-\infty}^{\infty} p(x)\,dx = 1$.

Well, choosing $u = -e^{-x}$, we get $du = e^{-x}dx$, so

$$\int p(x)\,dx = \int Ce^{-x}e^{-e^{-x}}\,dx = \int Ce^{u}\,du = Ce^{-e^{-x}} + K$$

for a constant K.

Now,

$$1 = \int_{-\infty}^{\infty} p(x)\,dx = \int_{-\infty}^{0} p(x)\,dx + \int_0^\infty p(x)\,dx = \lim_{R\to-\infty}\int_R^0 p(x)\,dx + \lim_{R\to\infty}\int_0^R p(x)\,dx$$

$$= \lim_{R\to-\infty} Ce^{-e^{-x}}\Big|_R^0 + \lim_{R\to\infty} Ce^{-e^{-x}}\Big|_0^R = Ce^{-e^0} - \lim_{R\to-\infty} Ce^{-e^{-R}} + \lim_{R\to\infty} Ce^{-e^{-R}} - Ce^{-e^0}$$

$$= 0 + C = C$$

So $C = 1$, and $p(x) = e^{-x}e^{-e^{-x}}$.

[NOTE: You MUST split this integral. See the definition in section 8.7, and consider Exercise 8.7.50.]

And

$$P(-4 \le X \le 4) = \int_{-4}^{4} p(x)\, dx = e^{-e^{-x}}\Big|_{-4}^{4} = e^{-e^{-4}} - e^{-e^{4}} \approx 0.982$$

8.8.6

Problem 8.8.8

Show that the density function $p(x) = \frac{1}{\pi}\frac{2}{x^2+1}$ on $[0, \infty)$ has infinite mean.

SOLUTION. Using the substitution $u = x^2 + 1$, we find that the mean of this density function is given by

$$\mu = \int_0^\infty xp(x)\, dx = \lim_{R\to\infty} \int_0^R \frac{1}{\pi}\frac{2x}{x^2+1}\, dx = \lim_{R\to\infty} \frac{1}{\pi} \ln|x^2+1|\Big|_0^R = \infty$$

So this integral diverges, so this density function has infinite mean. 8.8.8

Problem 8.8.22

*According to **Maxwell's Distribution Law**, in a gas of molecular mass m, the speed v of a molecule in a gas at temperature T (kelvins) is a random variable with density*

$$p(v) = 4\pi(\frac{m}{2\pi kT})^{3/2} v^2 e^{-mv^2/(2kT)} \qquad (v \ge 0)$$

where k is Boltzmann's constant. Show that the average molecular speed is equal to $(8kT/\pi m)^{1/2}$.

SOLUTION. We are asked to find the average speed, so we are taking the mean of this density function:

$$\mu = \int_0^\infty vp(v)\, dv = 4\pi(\frac{m}{2\pi kT})^{3/2} \int_0^\infty v^3 e^{-mv^2/(2kT)}\, dv$$

Let $\alpha = -m/(2kT)$. We'll compute the indefinite integral

$$\int v^3 e^{\alpha v^2}\, dv$$

To do this, we use integration by parts, with $u = v^2$, $dh = ve^{\alpha v^2}$. Then $du = 2v\, dv$ and $h = \frac{1}{2\alpha}e^{\alpha v^2}$, and

$$\int v^3 e^{\alpha v^2}\, dv = \frac{1}{2\alpha} v^2 e^{\alpha v^2} - \frac{1}{2\alpha}\int 2ve^{\alpha v^2}\, dv$$

To compute the latter integral, choose $u = \alpha v^2$, then $du = 2\alpha v dv$, so

$$\frac{1}{2\alpha}\int 2ve^{\alpha v^2}\ dv = \frac{1}{2\alpha^2}\int e^u\ du = \frac{1}{2\alpha^2}e^{\alpha v^2} - C$$

So

$$\int v^3 e^{\alpha v^2} dv = \frac{1}{2\alpha}v^2 e^{\alpha v^2} - \frac{1}{2\alpha^2}e^{\alpha v^2} + C$$

Thus

$$\begin{aligned}
\mu = \int_0^\infty vp(v)\ dv &= 4\pi(\frac{m}{2\pi kT})^{3/2}\int_0^\infty v^3 e^{-mv^2/(2kT)}\ dv \\
&= 4\pi(\frac{m}{2\pi kT})^{3/2}\lim_{R\to\infty}(\frac{1}{2\alpha}v^2 e^{\alpha v^2} - \frac{1}{2\alpha^2}e^{\alpha v^2})\Big|_0^R \\
&= 4\pi\frac{1}{2\alpha}(\frac{m}{2\pi kT})^{3/2}(\lim_{R\to\infty}[e^{R^2\alpha}(R^2 - 1/\alpha)] + 1/\alpha)
\end{aligned}$$

Recall that $\alpha = -m/(2kT) < 0$, so we can use L'Hopital's rule twice to evaluate the limit:

$$\lim_{R\to\infty} e^{R^2\alpha}(R^2 - 1/\alpha) = \lim_{R\to\infty}\frac{R^2 - 1/\alpha}{e^{-R^2\alpha}} = \lim_{R\to\infty}\frac{2R}{-2R\alpha e^{-R^2\alpha}} = \lim_{R\to\infty}\frac{-1}{\alpha e^{-R^2\alpha}} = 0$$

So

$$\begin{aligned}
\mu = 4\pi\frac{1}{2\alpha}(\frac{m}{2\pi kT})^{3/2}(\lim_{R\to\infty}[e^{R^2\alpha}(R^2 - 1/\alpha)] + 1/\alpha) &= 4\pi\frac{1}{2\alpha}(\frac{m}{2\pi kT})^{3/2}(\frac{1}{\alpha}) \\
= 2\pi\frac{1}{\alpha^2}(\frac{m}{2\pi kT})^{3/2} &= 2\pi(\frac{2kT}{m})^2(\frac{m}{2\pi kT})\sqrt{\frac{m}{2\pi kT}} \\
&= \frac{4kT}{m}\sqrt{\frac{m}{2\pi kT}} = \sqrt{\frac{8kT}{\pi m}}
\end{aligned}$$

8.8.22

Problem 8.8.28

Calculate μ *and* σ, *where* σ *is the* ***standard deviation***, *defined by*

$$\sigma^2 = \int_{-\infty}^\infty (x - \mu)^2 p(x)\ dx$$

Here $p(x) = \frac{1}{r}e^{-x/r}$ *on* $[0, \infty)$, *where* $r > 0$.

SOLUTION.

$$\mu = \int_0^\infty xp(x)\ dx = \lim_{R\to\infty}\int_0^R \frac{1}{r}xe^{-x/r}\ dx$$

Using integration by parts, with $u = x$ and $dv = \frac{1}{r}e^{-x/r}\ dx$, we get $du = dx$ and $v = -e^{-x/r}$, so

$$\mu = \lim_{R\to\infty} \int_0^R \frac{1}{r} x e^{-x/r}\, dx = -\lim_{R\to\infty} x e^{-x/r}\Big|_0^R + \lim_{R\to\infty} \int_0^R e^{-x/r}\, dx$$
$$= -\lim_{R\to\infty} Re^{-R/r} - \lim_{R\to\infty} re^{-x/r}\Big|_0^R = -\lim_{R\to\infty} \frac{R}{e^{R/r}} - \lim_{R\to\infty} re^{-R/r} + r$$

We can solve the first limit with L'Hopital's rule:

$$-\lim_{R\to\infty} \frac{R}{e^{R/r}} - \lim_{R\to\infty} re^{-R/r} + r = -\lim_{R\to\infty} \frac{1}{r^{-1}e^{R/r}} + 0 + r = 0 + r = r$$

So $\mu = r$

Now,

$$\sigma^2 = \int_0^\infty \frac{1}{r}(x-r)^2 e^{-x/r}\, dx = \frac{1}{r}\int_0^\infty (x^2 e^{-x/r} - 2rxe^{-x/r} + r^2 e^{-x/r})dx$$
$$= \frac{1}{r}\int_0^\infty x^2 e^{-x/r} dx - 2\int_0^\infty xe^{-x/r} dx + r\int_0^\infty e^{-x/r} dx$$

Now, the third integral can be rewritten

$$r\int_0^\infty e^{-x/r} dx = r^2 \int_0^\infty p(x) dx = r^2(1) = r^2$$

because $p(x)$ is a density function.

The second integral can be rewritten

$$-2\int_0^\infty xe^{-x/r} dx = -2r\mu = -2r^2$$

The first integral can be solved with integration by parts, using $u = x^2$ and $dv = e^{-x/r}dx$, so $du = 2xdx$ and $v = -re^{-x/r}$. Then

$$\frac{1}{r}\int_0^\infty x^2 e^{-x/r} dx = \frac{1}{r}\lim_{R\to\infty} x^2(-r)e^{-x/r}\Big|_0^R + \frac{1}{r}\int_0^\infty 2rxe^{-x/r} dx$$
$$= \frac{1}{r}\lim_{R\to\infty} \frac{R^2(-r)}{e^{-x/r}} + 2r\mu$$

The first limit is 0, by two applications of L'Hopital's rule, so we get

$$\frac{1}{r}\lim_{R\to\infty} \frac{R^2(-r)}{e^{-x/r}} + 2r\mu = 2r(r) = 2r^2$$

Putting these together, we find that $\sigma^2 = 2r^2 - 2r^2 + r^2 = r^2$, so $\sigma = r$. 8.8.28

Problem 8.9.2

Find M_4 *and* T_4 *for* $\int_0^4 \sqrt{x} dx$

SOLUTION. $\Delta x = (4-0)/4 = 1$, so $x_i = i$ for $0 \le i \le 4$

Thus

$$M_4 = \Delta x(\sqrt{x_0 + \frac{1}{2}\Delta x} + \sqrt{x_1 + \frac{1}{2}\Delta x} + \sqrt{x_2 + \frac{1}{2}\Delta x} + \sqrt{x_3 + \frac{1}{2}\Delta x})$$
$$= \sqrt{1/2} + \sqrt{3/2} + \sqrt{5/2} + \sqrt{7/2} \approx 5.38382$$

And

$$T_4 = \frac{1}{2}\Delta x(\sqrt{0} + 2\sqrt{1} + 2\sqrt{2} + 2\sqrt{3} + \sqrt{4}) \approx 5.14626$$

8.9.2

Problem 8.9.19

Find S_8 given by Simpson's Rule for $\int_1^4 \ln x \, dx$

SOLUTION. $\Delta x = (4-1)/8 = 3/8$, so

$$S_8 = \frac{1}{3}(\frac{3}{8})(\ln 1 + 4\ln(1 + \frac{3}{8}) + 2\ln(1 + 2(\frac{3}{8})) + 4\ln(1 + 3(\frac{3}{8})) + \ldots + 4\ln(1 + 7(\frac{3}{8})) + \ln 4)$$
$$\approx 2.54499$$

8.9.19

Problem 8.9.52

Let $J = \int_0^\infty e^{-x^2} dx$ and $J_N = \int_0^N e^{-x^2} dx$. Although e^{-x^2} has no elementary antiderivative, it is known that $J = \sqrt{\pi}/2$. Let T_N be the Nth trapezoidal approximation to J_N. Calculate T_4 and show that T_4 approximates J to three decimal places.

SOLUTION. T_4 is the 4th trapezoidal approximation to $\int_0^4 e^{-x^2}$, so $\Delta x = (4-0)/4 = 1$, and we get $x_i = i$ for $0 \le i \le 4$, so

$$T_4 = \frac{1}{2}\Delta x(e^{-0^2} + 2e^{-1^2} + 2e^{-2^2} + 2e^{-3^2} + e^{-4^2})$$

$$= \frac{1}{2}(1 + 2e^{-1} + 2e^{-4} + 2e^{-9} + e^{-16}) \approx .8863185$$

Then the error of the approximation of J by T_4 is given by

$$|J - T_4| = |\sqrt{\pi}/2 - .8863185| \approx |.8862269 - .8863185| < 10^{-3},$$

as desired.

8.9.52

Problem 8.9.64

Calculate M_{10} and S_{10} for the integral $\int_0^1 \sqrt{1-x^2} dx$, whose value we know to be $\pi/4$ (one-quarter of the area of the unit circle.

(a) *We usually expect S_N to be more accurate than M_N. Which of M_{10} and S_{10} is more accurate in this case?*

(b) *How do you explain the result of part (a)?*

SOLUTION. $\Delta x = (1-0)/10 = .1$, and $x_i = (.1)i$ for $0 \le i \le 10$, so

$$M_{10} = .1(\sqrt{1-.05^2} + \sqrt{1-.15^2} + ... + \sqrt{1-.85^2} + \sqrt{1-.95^2}) \approx .788103;$$

$$S_{10} = \frac{1}{3}(.1)(\sqrt{1-0^2} + 4\sqrt{1-.1^2} + 2\sqrt{1-.2^2} + ... + 2\sqrt{1-.8^2} + 4\sqrt{1-.9^2} + \sqrt{1-1})$$

$$\approx .781752$$

(a) The approximation error for M_{10} is $|\pi/4 - .788103| \approx .0027$. The approximation error for S_{10} is $|\pi/4 - .781752| \approx .00365$

So M_{10} is actually closer to $\int_0^1 \sqrt{1-x^2}dx$ in this case.

(b) Recall that our error bounds for M_N and S_N require that $|f''(x)|$ and $|f^{(4)}(x)|$, respectively, to be continuous and bounded on the interval $[a, b]$ (here $[a, b] = [0, 1]$ and $f(x) = \sqrt{1-x^2}$).

Computing the first four derivatives of f, we get

$$f'(x) = -x(1-x^2)^{-1/2}$$
$$f''(x) = -(1-x^2)^{-3/2}$$
$$f^{(3)}(x) = -3x(1-x^2)^{-5/2}$$
$$f^{(4)}(x) = -3(x^2+1)(1-x^2)^{(-7/2)}$$

Now, we see that as $x \to 1^-$, $|f''(x)|$ and $|f^{(4)}(x)|$ go to ∞, so our theorems about error bounds do not apply. However, we can see that as $x \to 1^-$, $|f^{(4)}(x)|$ goes to ∞ much faster than $|f''(x)|$, so the hypotheses on our error bound theorems are, in some sense, more violated in the case of S_N than in the case of M_N, here. Thus, it is not surprising that M_N would provide a better approximation.

8.9.64