Problem 9.1.3

Find the arc length of $y = \frac{1}{12}x^3 + x^{-1}$ *for* $1 \le x \le 2$.
Hint: Show that $1 + (y')^2 = \left(\frac{1}{4}x^2 + x^{-2}\right)^2$.

SOLUTION. $y' = \frac{x^2}{4} - x^{-2}$ so we have

$$(y')^2 + 1 = \left(\frac{x^2}{4} - x^{-2}\right)^2 + 1 = \frac{x^4}{16} - 2\frac{x^{-2}x^2}{4} + x^{-4} + 1 = \frac{x^4}{16} + \frac{1}{2} + x^{-4} = \left(\frac{x^2}{4} + x^{-2}\right)^2$$

as in the hint. So,

$$s = \int_1^2 \sqrt{1 + (y')^2}dx = \int_1^2 \sqrt{\left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2}dx = \int_1^2 \left|\frac{x^2}{4} + \frac{1}{x^2}\right| dx = \int_1^2 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx$$

since $\frac{x^2}{4} + \frac{1}{x^2} > 0$

Lastly,

$$s = \left(\frac{x^3}{12} - \frac{1}{x}\right)\Bigg|_1^2 = \frac{8}{12} - \frac{1}{2} - \frac{1}{12} + 1 = \frac{13}{12}$$

9.1.3

Problem 9.1.21

Find the value of a *such that the arc length of the catenary* $y = \cosh x$ *for* $-a \le x \le a$ *equals 10.*

SOLUTION. We find the arc length s of $y = \cosh x$ from $-a \le x \le a$ by

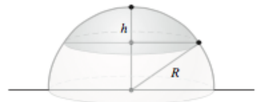
$$s = \int_{-a}^{a} \sqrt{1 + (y')^2}dx = \int_{-a}^{a} \sqrt{1 + (\sinh x)^2}dx = \int_{-a}^{a} \sqrt{(\cosh x)^2} = \int_{-a}^{a} \cosh x dx$$

$$= \sinh(x)\Big|_{-a}^{a} = \sinh(a) - \sinh(-a) = 2\sinh a$$

since sinh is odd. Setting $s = 10$ we see $2\sinh a = 10$ and $a = \text{arcsinh} 5$. 9.1.21

Problem 9.1.48

Show that the surface area of a spherical cap of height h *and radius* R *has surface area* $2\pi Rh$



SOLUTION. The equation of the circle of radius R centered at the origin is $x^2 + y^2 = R^2$. So, as part of the sphere centered at the origin, this cap can be obtained by rotating the right half of this circle about the y-axis from the bottom of the cap at $y = R - h$ to the top of the cap at $y = R$. The right half of the circle is given by equation $x = \sqrt{R^2 - y^2}$, so we have that the radii of the frustums are $x = \sqrt{R^2 - y^2}$, from $y = R - h$ to $y = R$. The length of the frustums are given by arc length

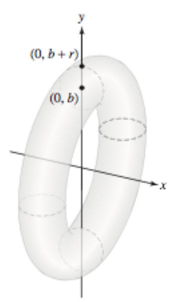
$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \sqrt{1 + \left(\frac{-2y}{2\sqrt{R^2 - y^2}}\right)^2} dy = \sqrt{1 + \frac{y^2}{R^2 - y^2}} dy = \sqrt{\frac{R^2}{R^2 - y^2}} dy$$

Thus, surface area is

$$\int_{y=R-h}^{R} 2\pi\sqrt{R^2 - y^2}\frac{R dy}{\sqrt{R^2 - y^2}} = 2\pi\int_{R-h}^{R} R dy = 2\pi(Ry)\Big|_{R-h}^{R} = 2\pi(R^2 - (R^2 - hR)) = 2\pi Rh$$

9.1.48

Problem 9.1.49 *Find the surface area of the torus obtained by rotating the circle* $x^2 + (y - b)^2 = r^2$ *about the* x*-axis*



SOLUTION. Solving for y we get $y = b \pm \sqrt{r^2 - x^2}$. So the top half of this circle is given by $y_1 = b + \sqrt{r^2 - x^2}$ and the bottom half of this circle given $y_2 = b - \sqrt{r^2 - x^2}$. Then the surface area of the torus is the surface area of both of these halves rotated about the x-axis from $x = -r$ to $x = r$:

$$\int_{-r}^{r} 2\pi\left(b + \sqrt{r^2 - x^2}\right)\sqrt{1 + (y_1')^2}dx + \int_{-r}^{r} 2\pi\left(b - \sqrt{r^2 - x^2}\right)\sqrt{1 + (y_2')^2}dx$$

$$= \int_{-r}^{r} 2\pi\left(b + \sqrt{r^2 - x^2}\right)\sqrt{1 + \left(\frac{-2x}{2\sqrt{r^2 - x^2}}\right)^2}dx + \int_{-r}^{r} 2\pi\left(b - \sqrt{r^2 - x^2}\right)\sqrt{1 + \left(\frac{2x}{2\sqrt{r^2 - x^2}}\right)^2}dx$$

$$= \int_{-r}^{r} 2\pi \left(b + \sqrt{r^2 - x^2}\right) \sqrt{1 + \frac{x^2}{(r^2 - x^2)}}\, dx + \int_{-r}^{r} 2\pi \left(b - \sqrt{r^2 - x^2}\right) \sqrt{1 + \frac{x^2}{(r^2 - x^2)}}\, dx$$

$$= \int_{-r}^{r} 2\pi \left(b + \sqrt{r^2 - x^2}\right) \sqrt{\frac{r^2}{(r^2 - x^2)}}\, dx + \int_{-r}^{r} 2\pi \left(b - \sqrt{r^2 - x^2}\right) \sqrt{\frac{r^2}{(r^2 - x^2)}}\, dx$$

$$= 2\pi \int_{-r}^{r} \sqrt{\frac{r^2}{(r^2 - x^2)}} \left(b + \sqrt{r^2 - x^2} + b - \sqrt{r^2 - x^2}\right) dx = 2\pi \int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}} (2b)\, dx$$

Letting $x = ur$ and $dx = r du$

$$= 4\pi rb \int_{-r}^{r} \frac{1}{\sqrt{r^2 - x^2}}\, dx = 4\pi rb \int_{-1}^{1} \frac{du}{\sqrt{1 - u^2}} = 4\pi rb \left(\arcsin u\right) \Big|_{-1}^{1}$$

$$= 4\pi rb \left(\arcsin(1) - \arcsin(-1)\right) = 4\pi rb \left(\frac{\pi}{2} - \frac{-\pi}{2}\right) = 4\pi^2 rb$$

9.1.49

Problem 9.4.4 *Calculate the Taylor polynomials T_2 and T_3 centered at $x = -1$ for the function $f(x) = \frac{1}{1+x^2}$*

SOLUTION. We calculate the derivatives:

$$f(x) = \frac{1}{1+x^2} \qquad f(-1) = \frac{1}{2} \tag{1}$$

$$f'(x) = -\frac{2x}{(1+x^2)^2} \qquad f'(-1) = \frac{1}{2} \tag{2}$$

$$f''(x) = -\frac{2}{(1+x^2)^2} + \frac{8x^2}{(1+x^2)^3} \qquad f''(-1) = -\frac{1}{2} + \frac{8}{8} = \frac{1}{2} \tag{3}$$

$$f'''(x) = \frac{8x}{(1+x^2)^3} + \frac{16x}{(1+x^2)^3} - \frac{48x^3}{(1+x^2)^4} \qquad f'''(-1) = 0 \tag{4}$$

Since $f'''(-1) = 0$, we see that

$$T_3 = \sum_{j=0}^{3} \frac{f^j(-1)}{j!}(x+1)^j = T_2 = \sum_{j=0}^{2} \frac{f^j(-1)}{j!}(x+1)^j = \frac{1}{2}+\frac{1}{2}(x+1)+\frac{1}{2(2!)}(x+1)^2 = \frac{1}{2}+\frac{1}{2}(x+1)+\frac{1}{4}(x+1)^2$$

9.4.4

Problem 9.4.8 *Calculate the Taylor polynomials T_2 and T_3 centered at $x = \frac{\pi}{4}$ for the function $f(x) = \tan x$*

SOLUTION. First, we calculate and evaluate the needed derivatives:

$$f(x) = \tan x \to f(\frac{\pi}{4}) = 1$$

$$f'(x) = \sec^2 x \to f'(\frac{\pi}{4}) = 2$$

$$f''(x) = 2\sec^2 x \tan x \to f''(\frac{\pi}{4}) = 4$$

$$f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x \to f'''(\frac{\pi}{4}) = 16$$

Then, we see

$$T_2(x) = \sum_{j=0}^{2} \frac{f^j(\frac{\pi}{4})}{j!}(x - \frac{\pi}{4})^j = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2$$

And

$$T_3(x) = \sum_{j=0}^{3} \frac{f^j(\frac{\pi}{4})}{j!}(x - \frac{\pi}{4})^j = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$$

9.4.8

Problem 9.4.18 *Show that the Maclaurin polyomials for $f(x) = \ln(1+x)$ are*

$$T_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n}$$

SOLUTION. Let $f(x) = \ln(1+x) \to f(0) = 0$. Then we see

$$f'(x) = (1+x)^{-1} \to f'(0) = 1$$
$$f''(x) = -(1+x)^{-2} \to f''(0) = -1$$
$$f'''(x) = 2(1+x)^{-3} \to f'''(0) = 2$$
$$f^{(4)}(x) = -3!(1+x)^{-4} \to f^{(4)}(0) = -6$$
$$f^{(5)}(x) = 4!(1+x)^{-5} \to f^{(5)}(0) = 24$$

So that in general

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n} \to f^{(n)}(0) = (-1)^{n-1}(n-1)!$$

Thus,

$$T_n(x) = x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \cdots + \frac{(-1)^{n-1}(n-1)!}{n!}x^n = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n}$$

9.4.18

Problem 9.4.59 *Let a be the length of the chord $\overline{AC}$ of angle θ of the unit circle. Derive the following approximation for the excess of the arc over the chord:*

$$\theta - a \approx \frac{\theta^3}{24}$$

Hint: Show that $\theta - a = \theta - 2\sin(\frac{\theta}{2})$ and use the third Maclaurin polynomial as an approximation.

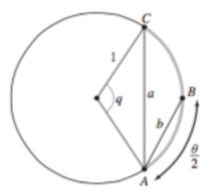


FIGURE 14 Unit circle.

SOLUTION. Draw a line from the center of the circle to B, forming (two) right triangles with hypotenuses length 1 and side opposite angle $\frac{\theta}{2}$ of length $\frac{a}{2}$. Thus, we see that $\sin\left(\frac{\theta}{2}\right) = \frac{a}{2}$.

Then, we obtain the hint: $2\sin\left(\frac{\theta}{2}\right) = a$ so $\theta - 2\sin\left(\frac{\theta}{2}\right) = \theta - a$. From the hint, we next approximate $f(\theta) = \sin\left(\frac{\theta}{2}\right)$ using the third Maclaurin polynomial:

$$f(\theta) = \sin\left(\frac{\theta}{2}\right) \qquad f(0) = 0 \tag{5}$$
$$f'(\theta) = \frac{1}{2}\cos\left(\frac{\theta}{2}\right) \qquad f'(0) = \frac{1}{2} \tag{6}$$
$$f''(\theta) = -\frac{1}{4}\sin\left(\frac{\theta}{2}\right) \qquad f''(0) = 0 \tag{7}$$
$$f'''(\theta) = -\frac{1}{8}\cos\left(\frac{\theta}{2}\right) \qquad f'''(0) = -\frac{1}{8} \tag{8}$$

Thus we have

$$\sin\left(\frac{\theta}{2}\right) = f(\theta) \approx \sum_{j=0}^{3} \frac{f^j(0)}{j!}\theta^j = 0 + \frac{1}{2}\theta + 0 + \left(-\frac{1}{8(3!)}\right)\theta^3 = \frac{1}{2}\theta - \frac{1}{48}\theta^3$$

And

$$\theta - a = \theta - 2\sin\left(\frac{\theta}{2}\right) \approx \theta - 2\left(\frac{1}{2}\theta - \frac{1}{48}\theta^3\right) = \theta - \theta + \frac{1}{24}\theta^3 = \frac{1}{24}\theta^3$$

9.4.59

Problem 10.1.14 *Use Separation of Variables to find the general solution of* $y' + 4xy^2 = 0$

SOLUTION.

$$\frac{dy}{dx} = -4xy^2 \to y^{-2}\ dy = -4x\ dx$$

Integrating both sides,

$$\int y^{-2}\ dy = -4\int x\ dx$$
$$-y^{-1} = -2x^2 + C$$
$$y^{-1} = 2x^2 + C$$

Thus we have that for arbitrary C,

$$y = \frac{1}{2x^2 + C}$$

10.1.14

Problem 10.1.16 *Use Separation of Variables to find the general solution of* $y' - e^{x+y} = 0$

SOLUTION.

$$\frac{dy}{dx} = e^x e^y \to e^{-y}\ dy = e^x\ dx$$

Integrating both sides,

$$\int e^{-y}\ dy = \int e^x\ dx$$
$$-e^{-y} = e^x + C$$
$$e^{-y} = -e^x + C$$

Thus for arbitrary C,

$$y = -\ln(-e^x + C)$$

10.1.16

Problem 10.1.40 *Solve the Initial Value Problem* $\frac{dy}{dt} = te^{-y}, y(1) = 0$

SOLUTION. We see that $e^y\,dy = t\,dt$, so integrating we obtain $e^y = \frac{t^2}{2} + C$. From $y(1) = 0$ we see $e^0 = \frac{1}{2} + C \to C = \frac{1}{2}$ Hence,

$$y = \ln\left(\frac{t^2}{2} + \frac{1}{2}\right)$$

10.1.40

Problem 10.1.45 *Find all values of a such that $y = x^a$ is a solution of $y'' - 12x^{-2}y = 0$*

SOLUTION. If $y = x^a$ then $y' = ax^{a-1}$ and $y'' = a(a-1)x^{a-2}$. So we see that we must have

$$(a(a-1)x^{a-2}) - 12x^{-2}(x^a) = (a(a-1)x^{a-2}) - 12x^{a-2} = (a(a-1) - 12)x^{a-2} = (a^2 - a - 12)x^{a-2} = 0$$

This can happen if and only if $(a^2 - a - 12) = (a-4)(a+3) = 0$. Thus, this is when $a = 4$ or $a = -3$.

10.1.45

Problem 10.1.55

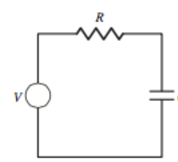


FIGURE 8 An *RC* circuit.

Figure 8 shows a circuit consisting of a resistor of R ohms, a capacitor of C farads, and a battery of voltage V. When the circuit is completed, the amount of charge $q(t)$ (in coulombs) on the plates of the capacitor varies according to the differential equation (t in seconds)

$$R\frac{dq}{dt} + \frac{1}{C}q = V$$

where R, C, and V are constants.

1. *Solve for $q(t)$, assuming that $q(0) = 0$.*
2. *Sketch the graph of q.*
3. *Show that $\lim_{t\to\infty} q(t) = CV$.*
4. *Show that the capacitor charges to approximately 63% of its final value CV after a time period of length $\tau = RC$ (τ is called the time constant of the capacitor).*

SOLUTION. 1.

$$R\frac{dq}{dt} + \frac{1}{C}q = V \text{ so } \frac{dq}{dt} = \frac{1}{R}\left(\frac{-1}{C}q + V\right) = \frac{-q}{RC} + \frac{V}{R} = \frac{-q + VC}{RC}$$

Rearranging,

$$\frac{dq}{-q+VC} = \frac{dt}{RC} \text{ and } \int \frac{dq}{-q+VC} = \int \frac{dt}{RC} \text{ so } -\ln|-q+CV| = \frac{t}{RC} + k$$

Then,

$$\ln|-q+CV| = -\frac{t}{RC} - k \text{ and } |-q+CV| = e^{-\frac{t}{RC}} e^{-k} = Ke^{-\frac{t}{RC}}$$

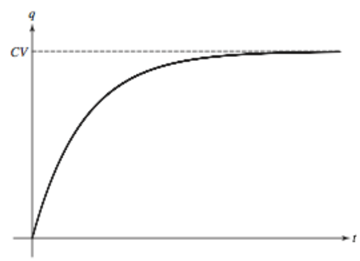
Thus,

$$-q+CV = \pm Ke^{-\frac{t}{RC}} = Ke^{-\frac{t}{RC}} \text{ so } q(t) = CV - Ke^{-\frac{t}{RC}}$$

Plugging in $q(0) = 0$, we see $0 = CV - K \to K = CV$. Thus, we have

$$q(t) = CV - CVe^{-\frac{t}{RC}}$$

2. The graph is below



3. From part a, $\lim_{t\to\infty} CV - CVe^{-\frac{t}{RC}} = CV - CV \lim_{t\to\infty} e^{-\frac{t}{RC}} = CV - CV(0) = CV$
4. After $\tau = RC$ we have $q(t) = q(RC) = CV - CVe^{-\frac{RC}{RC}} = CV(1-e^{-1}) \approx .632CV$.

10.1.55