HOMEWORK SOLUTIONS MATH 1910

Sections 11.1, 11.2, 11.3 Fall 2016

Problem 11.1.26 *Use Theorem 1 to determine the limit of the sequence or state that the sequence diverges.*

$$r_n = \ln n - \ln(n^2+1)$$

SOLUTION. We have $r_n = f(n)$, where $f(x) = \ln x - \ln(x^2+1)$; thus,

$$\lim_{n\to\infty}(\ln n - \ln(n^2+1)) = \lim_{n\to\infty}(\ln x - \ln(x^2+1)) = \lim_{x\to\infty} \ln \frac{x}{x^2+1}$$

but this function diverges as $x \to \infty$, so r_n diverges as well. $\boxed{11.1.26}$

Problem 11.1.74 *Show that* $a_n = \sqrt[3]{n+1} - n$ *is decreasing.*

SOLUTION. Let $f(x) = \sqrt[3]{x+1} - x$. Then

$$f'(x) = \frac{d}{dx}\left((x+1)^{1/3} - x\right) = \frac{1}{3}(x+1)^{-2/3} - 1$$

For $x \geq 1$,

$$\frac{1}{3}(x+1)^{-2/3} - 1 \leq \frac{1}{3}2^{-2/3} - 1 < 0$$

We conclude that f is decreasing on the interval $x \geq 1$; it follows that $a_n = f(n)$ is also decreasing. $\boxed{11.1.74}$

Problem 11.2.14 *Use partial fractions to rewrite* $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$ *as a telescoping series and find its sum.*

SOLUTION. By partial fraction decomposition

$$\frac{1}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3};$$

clearing denominators gives

$$1 = A(n+3) + Bn.$$

Setting $n = 0$ yields $A = \frac{1}{3}$, while setting $n = -3$ yields $B = \frac{1}{3}$. Thus,

$$\frac{1}{n(n+3)} = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right),$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right).$$

The general term in the sequence of partial sums for the series on the right-hand side is

$$S_N = \frac{1}{3}\left(1 - \frac{1}{4}\right) + \frac{1}{3}\left(\frac{1}{2} - \frac{1}{5}\right) + \frac{1}{3}\left(\frac{1}{3} - \frac{1}{6}\right) + \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7}\right) + \frac{1}{3}\left(\frac{1}{5} - \frac{1}{8}\right) + \frac{1}{3}\left(\frac{1}{6} - \frac{1}{9}\right)$$

$$+\cdots+\frac{1}{3}\left(\frac{1}{N-2}-\frac{1}{N+1}\right)+\frac{1}{3}\left(\frac{1}{N-1}-\frac{1}{N+2}\right)+\frac{1}{3}\left(\frac{1}{N}-\frac{1}{N+3}\right)$$

$$=\frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}\right)-\frac{1}{3}\left(\frac{1}{N+1}+\frac{1}{N+2}+\frac{1}{N+3}\right)=\frac{11}{18}-\frac{1}{3}\left(\frac{1}{N+1}+\frac{1}{N+2}+\frac{1}{N+3}\right).$$

Thus,

$$\lim_{N\to\infty} S_N=\lim_{N\to\infty}\left[\frac{11}{18}-\frac{1}{3}\left(\frac{1}{N+1}+\frac{1}{N+2}+\frac{1}{N+3}\right)\right]=\frac{11}{18},$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n(n+3)}=\frac{11}{18}.$$

$\boxed{11.2.14}$

Problem 11.2.16 *Find a formula for the partial sum S_N of $\sum\limits_{n=1}^{\infty}(-1)^{n-1}$ and show that the series diverges.*

SOLUTION. The partial sums of the series are:

$$\begin{aligned}
S_1 &= (-1)^{1-1} = 1;\\
S_2 &= (-1)^0 + (-1)^1 = 1-1 = 0;\\
S_3 &= (-1)^0 + (-1)^1 + (-1))^2 = 1;\\
S_4 &= (-1)^0 + (-1)^1 + (-1)^2 + (-1)^3 = 0; \ldots
\end{aligned}$$

In general,

$$S_N=\begin{cases}1 \text{ if N odd}\\ 0 \text{ if N even}\end{cases}$$

Because the values of S_N alterenate between 0 and 1, the sequence of partial sums diverges; this, in turn, implies that the series $\sum\limits_{n=1}^{\infty}(-1)^{n-1}$ diverges. $\boxed{11.2.16}$

Problem 11.2.18 *Use the nth Term Divergence Test (Theorem 3) to prove that the following series diverges:*

$$\sum_{n=1}^{\infty}\frac{n}{\sqrt{n^2+1}}.$$

SOLUTION. The general term, $\frac{n}{\sqrt{n^2+1}}$, has limit

$$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}}=\lim_{n\to\infty}\sqrt{\frac{n^2}{n^2+1}}=\lim_{n\to\infty}\sqrt{\frac{1}{1+(1/n^2)}}=1$$

Since the general term does not tend to zero, the series diverges. $\boxed{11.2.18}$

Problem 11.2.24 *Use the formula for the sum of a geometric series to find the sum or state that the series diverges.*

$$\frac{4^3}{5^3} + \frac{4^4}{5^4} + \frac{4^5}{5^5} + \frac{4^5}{5^5} + \cdots$$

SOLUTION. This a geometric series with

$$c = \frac{4^3}{5^3} \text{ and } r = \frac{4}{5}$$

so its sum is

$$\frac{c}{1-r} = \frac{4^3/5^3}{1-\frac{4}{5}} = \frac{4^3}{5^3 - 4 * 5^2} = \frac{64}{25}$$

$\boxed{11.2.24}$

Problem 11.2.26 *Use the formula for the sum of a geometric series to find the sum or state that the series diverges.*

$$\sum_{n=3}^{\infty} \left(\frac{3}{11}\right)^{-n}$$

SOLUTION. Rewrite this series as

$$\sum_{n=3}^{\infty} \left(\frac{11}{3}\right)^{n}$$

This is a geometric series with $r = \frac{11}{3} > 1$, so it is divergent. $\boxed{11.2.26}$

Problem 11.2.38 *Determine a reduced fraction that has this repeating decimal.*

$$0.454545\ldots$$

SOLUTION. The decimal may be regarded as a geometric series:

$$0.454545\ldots = \frac{45}{100} + \frac{45}{10000} + \frac{45}{1000000} + \cdots = \sum_{n=1}^{\infty} \frac{45}{10^{2n}}.$$

The series has first term $\frac{45}{100} = \frac{9}{20}$ and ratio $\frac{1}{100}$, so its sum is

$$0.45454545\ldots = \frac{9/20}{1-1/100} = \frac{9}{20} \cdot \frac{100}{99} = \frac{5}{11}.$$

$\boxed{11.2.38}$

Problem 11.2.47 *Give a counterexample to show that each of the following statements is false*

(a) *If the general term a_n tends to zero, then $\sum_{n=1}^{\infty} a_n = 0$.*

(b) *The Nth partial sum of the infinite series defined by a_n is a_N.*

(c) *If a_n tends to zero, then $\sum_{n=1}^{\infty} a_n$ converges.*

(d) *If a_n tends to L, then $\sum_{n=1}^{\infty} a_n = L$.*

SOLUTION. (a) If the general term an tends to zero, the series may or may not converge. Even if the series converges, it may not converge to zero. For example, with the harmonic series, where $a_n = \frac{1}{n}$, we have $\lim_{n\to\infty} a_n = 0$, but $\sum_{n=1}^{\infty} a_n$ diverges.

(b) The Nth partial sum of the series $\sum_{n=1}^{\infty} a_n$ is $S_N = a_1 + a_2 + \cdots + a_N$. For example, take the infinite series defined by $a_n = 1$ for all n. Then $a_N = 1$ but the Nth partial sum is N.

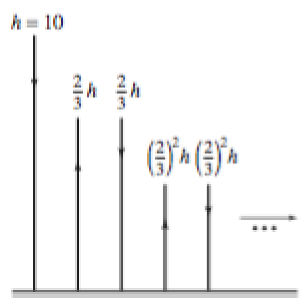
(c) If the general term a_n tends to zero, the series may or may not converge. See part (a) for an example.

(d) If $L \neq 0$, then the series diverges by the nth Term Divergence Test. If $L = 0$, the series may or may not converge, and even if it does converge, it may not converge to $L = 0$. For an example of the first case, take the series defined by $a_n = 1$ for all n; for the second, take the harmonic series.

$\boxed{11.2.47}$

Problem 11.2.56 *A ball dropped from a height of 10 ft begins to bounce vertically. Each time it strikes the ground, it returns to two-thirds of its previous height. What is the total vertical distance traveled by the ball if it bounces infinitely many times?*

SOLUTION. The distance traveled by the ball is shown in the accompanying figure:



The total distance d traveled by the ball is given by the following infinite sum:

$$d = h + 2\cdot\frac{2}{3}h + 2\cdot\left(\frac{2}{3}\right)^2 h + 2\cdot\left(\frac{2}{3}\right)^3 h + \cdots = h + 2h\left(\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots\right)$$
$$= h + 2h\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n.$$

We use the formula for the sum of a geometric series to compute the sum of the resulting series:

$$d = h + 2h\cdot\frac{(2/3)^1}{1-(2/3)} = h + 2h(2) = 5h.$$

With h = 10 feet, it follows that the total distance traveled by the ball is 50 feet. $\boxed{11.2.56}$

Problem 11.3.12 *Use the Integral Test to determine whether the infinite series* $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ *is convergent.*

SOLUTION. Let $f(x) = \frac{\ln x}{x^2}$. Because

$$f'(x) = \frac{\frac{1}{x}x^2 - 2x\ln x}{x^4} = \frac{x(1-2\ln x)}{x^4} = \frac{1-2\ln x}{x^3}$$

we see that $f'(x) < 0$ for $x > \sqrt{e} \sim 1.65$. We conclude that f is decreasing on the interval $x \geq 2$. Since f is also positive and continuous on this interval, the Integral Test can be applied. By Integration by Parts, we find

$$\int \frac{\ln x}{x^2} = -\frac{\ln x}{x} + \int x^{-2}dx = -\frac{\ln x}{x} - \frac{1}{x} + C$$

Therefore,

$$\int_2^\infty \frac{\ln x}{x^2} = \lim_{R\to\infty} \int_2^R \frac{\ln x}{x^2} = \lim_{R\to\infty} \frac{1}{2}\frac{\ln 2}{2} - \frac{1}{R} - \frac{\ln R}{R} = \frac{1+\ln 2}{2} - \lim_{R\to\infty} \frac{\ln R}{R}$$

We compute the resulting limit using L'Hopital's Rule:

$$\lim_{R\to\infty} \frac{\ln R}{R} = \lim_{R\to\infty} \frac{1/R}{1} = 0$$

Hence,

$$\int_2^\infty \frac{\ln x}{x^2} = \frac{1+\ln 2}{2}$$

This integral converges; therefore, the series $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ also converges. Since the convergence of the series is not affected by adding $\frac{\ln 1}{1}$, the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ also converges. $\boxed{11.3.12}$

Problem 11.3.24 *Use the Direct Comparison Test to determine whether the infinite series* $\sum_{n=4}^{\infty} \frac{\sqrt{n}}{n-3}$ *is convergent.*

SOLUTION. For $n \geq 4$, $\frac{\sqrt{n}}{n-3} \geq \frac{\sqrt{n}}{n} = \frac{1}{n^{1/2}}$.

The series $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a p-series with $p = 1/2 < 1$¡ so it diverges, and it continues to diverge if we drop the terms $n = 1, 2, 3$; that is, $\sum_{n=4}^{\infty} \frac{1}{n^{1/2}}$ also diverges.

By the Direct Comparison Test we can therefore conclude that the series $\sum_{n=4}^{\infty} \frac{\sqrt{n}}{n-3}$ diverges. $\boxed{11.3.24}$

Problem 11.3.42 *Use the Limit Comparison Test to prove convergence of divergence if the infinite series* $\sum_{n=2}^{\infty} \frac{n^3}{\sqrt{n^7+2n^2+1}}$.

SOLUTION. Let a_n be the general term of our series. Observe that

$$a_n = \frac{n^3}{\sqrt{n^7+2n^2+1}} = \frac{n^{-3}n^3}{n^{-3}\sqrt{n^7+2n^2+1}} = \frac{1}{\sqrt{n+2n^{-4}+n^{-6}}}$$

This suggests that we can compare our series with $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$.

The ratio of the terms is

$$\frac{a_n}{b_n} = \frac{1}{\sqrt{n+2n^{-4}+n^{-6}}}\frac{\sqrt{n}}{1} = \frac{1}{\sqrt{1+2n^{-5}+n^{-7}}}$$

Hence,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+2n^{-5}+n^{-7}}} = 1$$

The p-series $\sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$ diverges since $p = 1/2 < 1$. Therefore, our original series diverges. $\boxed{11.3.42}$

Problem 11.3.44 *Use the Limit Comparison Test to prove convergence or divergence of the infinite series*

$$\sum_{n=1}^{\infty} \frac{e^n+n}{e^{2n}-n^2}$$

SOLUTION. Let

$$a_n = \frac{e^n+n}{e^{2n}-n^2} = \frac{e^n+n}{(e^n-n)(e^n+n)} = \frac{1}{e^n-n}.$$

For large n,

$$\frac{1}{e^n-n} \approx \frac{1}{e^n} = e^{-n},$$

so we apply the Limit Comparison Test with $b_n = e^{-n}$. We find

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{e^n-n}}{e^{-n}} = \lim_{n\to\infty} \frac{e^n}{e^n-n} = 1.$$

The series $\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ is a geometric series with $r = \frac{1}{e} < 1$, so it converges. Because L exists, by the Limit Comparison Test we can conclude that the series $\sum_{n=1}^{\infty} \frac{e^n+n}{e^{2n}-n^2}$ also converges. $\boxed{11.3.44}$

Problem 11.3.54 *Determine convergence or divergence using any method covered so far*

$$\sum_{n=1}^{\infty} \frac{1}{n^2+\sin n}$$

SOLUTION. Apply the Limit Comparison Test with $a_n = \frac{1}{n^2+\sin n}$ and $b_n = \frac{1}{n^2}$;

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2+\sin n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{1+\frac{\sin n}{n^2}} = 1.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series. Because L exists, by the Limit Comparison Test we can conclude that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+\sin n}$ also converges. $\boxed{11.3.54}$

Problem 11.3.70 *Determine convergence or divergence using any method covered so far*

$$\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$$

SOLUTION. Apply the Limit Comparison Test with $a_n = \frac{\sin(1/n)}{\sqrt{n}}$ and $b_n = \frac{1/n}{\sqrt{n}}$:

$$L = \lim_{n\to\infty} \frac{\sin(1/n)}{\sqrt{n}} \cdot \frac{\sqrt{n}}{1/n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{u\to 0} \frac{\sin u}{u} = 1$$

so that $\sum a_n$ and $\sum b_n$ either both converge or both diverge. But

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1/n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

is a convergent p-series. Thus $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$ converges as well. $\boxed{11.3.70}$