Homework Solutions

Sections 5.3, 5.4, 5.5

MATH 1910

Fall 2016

Problem 5.3.24 *Evaluate the indefinite integral*

$$\int \frac{12 - z}{\sqrt{z}} dz.$$

Solution.

$$\int \frac{12 - z}{\sqrt{z}} dz = \int (12z^{-1/2} - z^{-1/2}) dz = 24z^{1/2} - \frac{2}{3}z^{3/2} + C$$

5.3.24

Problem 5.3.38 *Evaluate the indefinite integral*

$$\int \sec(x+5)\tan(x+5)dx.$$

Solution.

$$\int \sec(x+5)\tan(x+5)dx = \sec(x+5) + C$$

5.3.38

Problem 5.3.52

Solve the initial value problem: $\frac{dz}{dt} = t^{-3/2}$, $z(4) = -1$.

Solution. Integrate to find z:

$$z = \int t^{-3/2}\, dt = -2t^{-1/2} + C$$

and then use the initial condition $z(4) = -1$ to find z:

$$z(4) = -2(4)^{-1/2} + C = -1 \quad \Rightarrow C = 0.$$

Hence the solution is

$$z = -2t^{-1/2}.$$

5.3.52

Problem 5.3.56

Solve the initial value problem $\frac{dy}{dz} = \sin 2z$, $y\left(\frac{\pi}{4}\right) = 4$.

SOLUTION. Since $\frac{dy}{dz} = \sin 2z$, we have

$$y = \int \sin 2z dz = -\frac{1}{2}\cos 2z + C.$$

Thus

$$4 = y\left(\frac{\pi}{4}\right) = 0 + C$$

so that $C = 4$. Therefore, $y = 4 - \frac{1}{2}\cos 2z$. 5.3.56

Problem 5.3.68

$f''(\theta) = \cos\theta$, $f'\left(\frac{\pi}{2}\right) = 1$, $f\left(\frac{\pi}{2}\right) = 6$.

SOLUTION. Let $g(\theta) = f'(\theta)$. The problem statement gives

$$g'(\theta) = \cos\theta, \qquad g(\frac{\pi}{2}) = 1.$$

From $g'(\theta)$ we get $g(\theta) = \sin\theta + C$. From $g(\frac{\pi}{2}) = 1$ we get $1 + C = 1$, so $C = 0$. Hence, $f'(\theta) = g(\theta) = \sin\theta$. From $f'(\theta)$ we get $f(\theta) = -\cos\theta + C$. From $f(\frac{\pi}{2}) = 6$ we get $C = 6$, so

$$f(\theta) = -\cos\theta + 6.$$

5.3.68

Problem 5.3.76

Beginning at $t = 0$ (s) with initial velocity 4 (m/s), a particle moves in a straight line with acceleration $a(t) = 3t^{1/2}$ (m/s^2). Find the distance traveled after 25 (s).

SOLUTION. Integrate $a(t)$ to find the velocity $v(t)$:

$$v(t) = \int 3t^{1/2}\, dt = 2t^{3/2} + C,$$

then use the initial condition $v(0) = 4$ to solve for C:

$$v(0) = 2(0)^{/2} + C = 4 \quad \Rightarrow C = 4.$$

Hence $v(t) = 2t^{3/2} + 4$. Next, integrate $v(t)$ to find the displacement $s(t)$:

$$s(t) = \int (2t^{3/2} + 4)\, dt = \frac{4}{5}t^{5/2} + 4t + D.$$

We may assume that initial displacement is zero, and so we can solve for D:

$$s(0) = \frac{4}{5}(0)^{5/2} + 4(0) + D \quad \Rightarrow D = 0.$$

Finally, the distance traveled after 25 seconds is

$$s(25) = \frac{4}{5}(25)^{5/2} + 4(25) = 2600 \text{ (m)}.$$

5.3.76

Problem 5.3.81

Verify the linearity properties of the indefinite integral stated in Theorem 3.

SOLUTION. To verify the Sum Rule, let $F(x)$ and $G(x)$ be any antiderivatives of $f(x)$ and $g(x)$, respectively. Because

$$\frac{d}{dx}(F(x)+G(x)) = \frac{d}{dx}F(x) + \frac{d}{dx}G(x) = f(x)+g(x),$$

it follows that $F(x)+G(x)$ is an antiderivative of $f(x)+g(x)$; i.e.,

$$\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx.$$

To verify the Multiples Rule, again let $F(x)$ be any antiderivative of $f(x)$ and let c be a constant. Because

$$\frac{d}{dx}(cF(x)) = c\frac{d}{dx}F(x) = cf(x)$$

it follows that $cF(x)$ is an antiderivative of $cf(x)$; i.e.,

$$\int (cf(x))dx = c\int f(x)dx.$$

5.3.81

Problem 5.4.24

Evaluate the integral using FTC I:

$$\int_{8/27}^{1} \frac{10t^{4/3}-8t^{1/3}}{t^2}dt$$

SOLUTION.

$$\int_{8/27}^{1} \frac{10t^{4/3}-8t^{1/3}}{t^2}dt$$

=

$$\int_{8/27}^{1} (10t^{-2/3}-8t^{-5/3})dt = 30t^{1/3}+12t^{-2/3}\Big|_{8/27}^{1} = (30+12)-(20+27) = 5$$

5.4.24

Problem 5.4.36

Write the integral as a sum of integrals without absolute values and evaluate:

$$\int_0^3 \left|x^2-1\right| dx$$

SOLUTION.

$$\int_0^3 \left|x^2-1\right| dx = \int_0^1 (1-x^2)dx = \int_1^3 (x^2-1)dx = (x-\frac{x^3}{3})\Big|_0^1 + (\frac{x^3}{3}-x)\Big|_1^3 = (1-\frac{1}{3})-0+(9-3)-(\frac{1}{3}-1) = \frac{22}{3}$$

5.4.36

Problem 5.4.50

Show that the area of the shaded parabolic arch in Figure 7 is equal to four-thirds the area of the triangle shown.

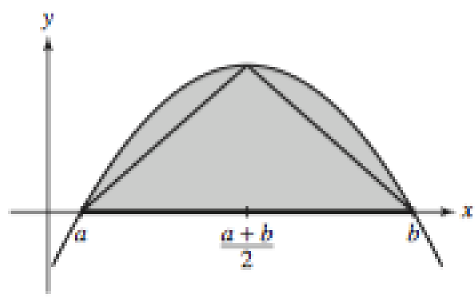

FIGURE 7 Graph of $y = (x - a)(b - x)$.

SOLUTION. We first calculate the area of the parabolic arch:

$$\begin{aligned}
\int_a^b (x-a)(b-x)dx &= -\int_a^b (x-a)(x-b)dx \\
&= -\int_a^b (x^2 - ax - bx + ab)dx \\
&= -\left(\frac{1}{3}x^3 - \frac{a}{2}x^2 - \frac{b}{2}x^2 + abx\right)\Big|_a^b \\
&= -\frac{1}{6}\left(2x^3 - 3ax^2 - 3bx^2 + 6abx\right)\Big|_a^b \\
&= -\frac{1}{6}\left((2b^3 - 3ab^2 - 3b^3 + 6ab^2) - (2a^3 - 3a^3 - 3ba^2 + 6a^2b)\right) \\
&= -\frac{1}{6}\left((-b^3 + 3ab^2) - (-a^3 + 3a^2b)\right) \\
&= -\frac{1}{6}\left(a^3 + 3ab^2 - 3a^2b - b^3\right) \\
&= \frac{1}{6}(b-a)^3
\end{aligned}$$

The indicated triangle has a base of length $b - a$ and a height of

$$\left(\frac{a+b}{2} - a\right)\left(b - \frac{a+b}{2}\right) = \left(\frac{b-a}{2}\right)^2.$$

Thus, the area of the triangle is

$$\frac{1}{2}(b-a)\left(\frac{b-a}{2}\right)^2 = \frac{1}{8}(b-a)^3.$$

Finally, we note that

$$\frac{1}{6}(b-a)^3 = \frac{4}{3}\cdot\frac{1}{8}(b-a)^3,$$

as required. 5.4.50

Problem 5.4.52

(a) Apply the Comparison Theorem (Theorem 5 in Section 5.2) to the inequality $\sin x \le x$ *(valid for all* $x \ge 0$*) to prove that*

$$1 - \frac{x^2}{2} \le \cos x \le 1.$$

(b) Apply it again to prove that

$$x - \frac{x^3}{6} \le \sin x \le x \quad (\textit{for } x \ge 0).$$

(c) Verify these inqualities for $x = 0.3$.

SOLUTION.

(a) We have

$$\int_0^x \sin t \, dt = -\cos t\big|_0^x = -\cos x + 1$$

and

$$\int_0^x t dt = \frac{1}{2}t^2\big|_0^x = \frac{1}{2}x^2.$$

Hence,

$$-\cos x + 1 \le \frac{x^2}{2}.$$

Solving, this gives $\cos x \ge 1 - \frac{x^2}{2}$. Then $\cos x \le 1$ follows automatically.

(b) The previous part gives us $1 - \frac{t^2}{2} \le \cos t \le 1$, for $t > 0$. Theorem 5 gives us, after integrating over the interval $[0, x]$,

$$x - \frac{x^3}{6} \le \sin x \le x.$$

(c) Substituting $x = 0.3$ into the inequalities obtained in (a) and (b) yeilds

$$0.955 \le 0.955336489 \le 1 \quad \text{and} \quad 0.2955 \le 0.2955202069 \le 0.3,$$

respectively. 5.4.52

Problem 5.5.1

Write the area function of $f(x) = 2x + 4$ with lower limit $a = -2$ as an integral and find a formula for it.

SOLUTION. Let $f(x) = 2x + 4$. The area function with lower limit $a = -2$ is

$$A(x) = \int_a^x f(t)dt = \int_{-2}^x (2t + 4)dt.$$

Carrying out the integration, we find

$$\int_{-2}^x (2t + 4)dt = (t^2 + 4t)\Big|_{-2}^x = (x^2 + 4x) - ((-2)^2 + 4(-2)) = x^2 + 4x + 4 = (x + 2)^2$$

Therefore, $A(x) = (x + 2)^2$

5.5.1

Problem 5.5.3

Let $G(x) = \int_1^x (t^2 - 2)dt$. Calculate $G(1)$, $G'(1)$, and $G'(2)$. Then find a formula for $G(x)$.

SOLUTION. Let $G(x) = \int_1^x (t^2 - 2)dt$. Then $G(1) = \int_1^1 (t^2 - 2)dt = 0$. Moreover, $G'(x) = x^2 - 2$, so that $G'(1) = -1$ and $G'(2) = 2$. Finally,

$$G(x) = \int_1^x (t^2 - 2)dt = (\frac{t^3}{3} - 2x)\Big|_1^x = (\frac{x^3}{3} - 2x) - (\frac{1}{3} - 2) = \frac{1}{3}x^3 - 2x + \frac{5}{3}$$

5.5.3

Problem 5.5.30 *Calculate the derivative* $\dfrac{d}{dx}\displaystyle\int_1^{1/x} \cos^3 t\, dt$.

SOLUTION. By the Chain Rule and the FTC,

$$\frac{d}{dx}\int_1^{1/x} \cos^3 t\, dt = \cos^3\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{x^2}\cos^3\left(\frac{1}{x}\right).$$

5.5.30

Problem 5.5.32

Calculate the derivative: $\frac{d}{dx}\int_{x^2}^{x^4} \sqrt{t}dt$.

SOLUTION. Let

$$F(x) = \int_{x^2}^{x^4} \sqrt{t}dt = \int_0^{x^4} \sqrt{t}dt - \int_0^{x^2} \sqrt{t}dt.$$

Applying the Chain Rule combined with FTC, we have

$$F'(x) = \sqrt{(x^4)}.4x^3 - \sqrt{(x^2)}.2x = 4x^5 - 2x\,|x|\,.$$

5.5.32

Problem 5.5.44

Match the property of A *with the corresponding property of the graph of* f. *Assume* f *is differentiable.*

Area function A

(a) A *is decreasing.*

(b) A *has a local maximum.*

(c) A *is concave up.*

(d) A *goes from concave up to concave down.*

Graph of f

(i) Lies below the x*-axis.*

(ii) Crosses the x*-axis from positive to negative.*

(iii) Has a local maximum.

(iv) f *is increasing.*

SOLUTION. Let $A(x) = \int_a^x f(t)dt$ be an area function of f. Then $A'(x) = f(x)$ and $A''(x) = f'(x)$.

(a) A is decreasing when $A'(x) = f(x) < 0$, that is, when f lies below the x-axis. This is choice (i).

(b) A has a local maximum at x_0 when $A'(x) = f(x)$ changes sign from $+$ to 0 to $-$ as x increases through x_0, that is, when f crosses the x-axis from positive to negative. This is choice (ii).

(c) A is concave up when $A''(x) = f'(x) > 0$, that is, when f is increasing. This corresponds to choice (iv).

(d) A goes from concave up to concave down at x_0 when $A''(x) = f'(x)$ changes sign from $+$ to 0 to 0 as x increases through x_0, that is, when f has a local maximum at x_0. This is choice (iii).

5.5.44

Problem 5.5.48

Figure 13 shows the graph of $f(x) = x\sin x$. *Let* $F(x) = \int_0^x t\sin t\,dt$.

(a) Locate the local max and absolute max of F *on* $[0, 3\pi]$.

(b) Justify graphically: F *has precisely one zero in* $[\pi, 2\pi]$.

(c) How many zeros does F *have in* $[0, 3\pi]$*?*

(d) Find the inflection points of F *on* $[0, 3\pi]$. *For each one, state whether the concavity changes from up to down or from down to up.*

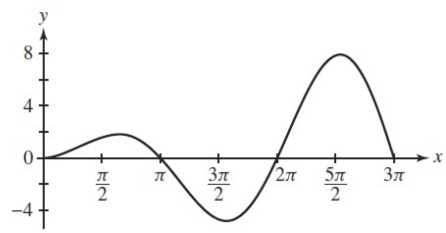

FIGURE 13 Graph of $f(x) = x \sin x$.

SOLUTION. Let $F(x) = \int_0^x t\sin t\,dt$. A graph of $f(x) = x\sin x$ is depicted in Figure 13. Note that $F'(x) = f(x)$ and $F''(x) = f'(x)$.

(a) For F to have a local maximum at $x_0 \in (0, 3\pi)$ we must have $F'(x_0) = f(x_0) = 0$ and $F' = f$ must change sign from positive to 0 to negative as x increases through x_0. This occurs at $x = \pi$. The absolute maximum of F on $[0, 3\pi]$ occurs at $x = 3\pi$ since (from the figure) the signed area between $x = 0$ and $x = c$ is greatest for $x = c = 3\pi$.

(b) At $x = \pi$, the value of F is positive since $f(x) > 0$ on $(0, \pi)$. As x increases along the interval $[\pi, 2\pi]$, we see that F decreases as the negatively signed area accumulates. Eventually the additional negatively signed area "outweighs" the prior positively signed area and F attains the value 0, say at $b \in (\pi, 2\pi)$. Thereafter, on $(b, 2\pi)$, we see that f is negative and thus F becomes and continues to be negative as the negatively signed area accumulates. Therefore, F takes the value 0 exactly once in the interval $[\pi, 2\pi]$.

(c) F has two zeroes in $[0, 3\pi]$. One is described in part (b) and the other must occur in the interval $[2\pi, 3\pi]$ because $F(x) < 0$ at $x = 2\pi$ but clearly the positively signed area over $[2\pi, 3\pi]$ is greater than the previous negatively signed area.

(d) Since f is differentiable, we have that F is twice differentiable on I. Thus F has an inflection point at x_0 provided $F''(x_0) = f'(x_0) = 0$ and $F''(x) = f'(x)$ changes sign at x_0. If $F'' = f'$ changes sign from positive to 0 to negative at x_0, then f has a local maximum at x_0. There is clearly such a value x_0 in the figure in the interval $[\pi/2, \pi]$ and another around $5\pi/2$. Accordingly, F has two inflection points where F changes from concave up to concave down. If $F'' = f'$ changes sign from negative to 0 to positive at x_0, then f has a local minimum at x_0. From the figure, there is such an x_0 around $3\pi/2$; so F has one inflection point where $F(x)$ changes from concave down to concave up.

5.5.48