Problem 5.6.6 *Find the displacement over the time interval* $[1, 6]$ *of a helicopter whose vertical velocity at time* t *is* $v(t) = 0.02t^2 + t$ *m/s.*

SOLUTION. Given $v(t) = \frac{1}{50}t^2 + t$ m/s, the change in height over $[1, 6]$ is

$$\int_1^6 v(t)dt = \int_1^6 \left(\frac{1}{50}t^2 + t\right) dt = \left(\frac{1}{150}t^3 + \frac{1}{2}t^2\right)\Big|_1^6 = \frac{284}{15} \approx 18.93\text{m}.$$

5.6.6

Problem 5.6.22

Figure 6 shows the migration rate $M(t)$ *of Ireland in the period* 1988 *to* 1998. *This is the rate at which people (in thousands per year) move into or out of the country.*

(a) Is the following integral positive or negative? What does this quantity represent?

$$\int_{1988}^{1998} M(t)dt$$

(b) Did migration in the period 1988 to 1998 result in a net influx of people into Ireland or a net outflow of people from Ireland?

(c) During which two years could the Irish prime minister announce, "We have hit an inflection point. We are still losing population, but the trend is now improving."

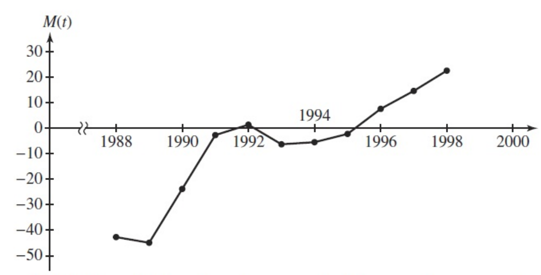

FIGURE 6 Irish migration rate (in thousands per year).

SOLUTION. (a) Because there appears to be more area below the t-axis than above in Figure 6,

$$\int_{1988}^{1998} M(t)dt$$

is negative. This quantity represents the net migration from Ireland during the period 1988 to 1998.

(b) As noted in part (a), there appears to be more area below the t-axis than above in Figure 6, so migration in the period 1988 to 1998 resulted in a net outflow of people from Ireland.

(c) The prime minister can make this statement when the graph of M is at a local minimum, which appears to be in the years 1989 and 1993.

5.6.22

Problem 5.6.26

Cardiac output is the rate R of volume of blood pumped by the heart per unit time (in liters per minute). Doctors measure R by injecting A mg of dye into a vein leading into the heart at $t = 0$ and recording the concentration $c(t)$ of dye (in milligrams per liter) pumped out at short regular time intervals (Figure 8).

(a) Explain: The quantity of dye pumped out in a small time interval $[t, t + \Delta t]$ is approximately $Rc(t)\Delta t$.

(b) Show that $A = R\int_0^T c(t)dt$, where T is large enough that all of the dye is pumped through the heart but not so large that the dye returns by recirculation.

(c) Assume $A = 5$ mg. Estimate R using the following values of $c(t)$ recorded at 1-second intervals from $t = 0$ to $t = 10$:

$$0, 0.4, 2.8, 6.5, 9.8, 8.9, 6.1, 4, 2.3, 1.1, 0$$

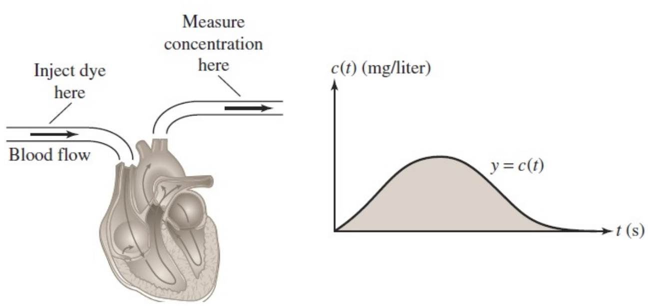

FIGURE 8

SOLUTION. (a) Over a short time interval, $c(t)$ is nearly constant. $Rc(t)$ is the rate of volume of dye (amount of fluid x concentration of dye in fluid) flowing out of the heart (in mg per minute). Over the short time interval $[t, t+\Delta t]$, the rate of flow of dye is approximately constant at $Rc(t)$ mg/minute. Therefore, the flow of dye over the interval is approximately $Rc(t)\Delta t$ mg.

(b) The rate of flow of dye is $Rc(t)$. Therefore the net flow between time $t = 0$ and time $t = T$ is

$$\int_0^T Rc(t)dt = R\int_0^T c(t)dt$$

If T is great enough that all of the dye is pumped through the heart, the net flow is equal to all of the dye, so

$$A = R\int_0^T c(t)dt$$

(c) In the table, $\Delta t = \frac{1}{60}$ minute, and $N = 10$. The right and left hand approximations of $\int_0^T c(t)dt$ are:

$$R_{10} = \frac{1}{60}(0.4 + 2.8 + 6.5 + 9.8 + 8.9 + 6.1 + 4 + 2.3 + 1.1 + 0) = 0.6983\frac{mg.minute}{liter}$$

$$L_{10} = \frac{1}{60}(0 + 0.4 + 2.8 + 6.5 + 9.8 + 8.9 + 6.1 + 4 + 2.3 + 1.1) = 0.6983\frac{mg.minute}{liter}$$

Both L_N and R_N are the same, so the average of L_N and R_N is 0.6983. Hence,

$$A = R\int_o^T c(t)dt$$

$$5mg = R(0.6983\frac{mg.minute}{liter})$$

$$R = \frac{5}{0.6983}\frac{liters}{minute} = 7.16\frac{liters}{minute}$$

5.6.26

Problem 5.6.29

Show that a particle, located at the origin at $t = 1$ and moving along the x-axis with velocity $v(t) = t^{-2}$ will never pass the point $x = 2$.

SOLUTION. The particle's velocity is $v(t) = s'(t) = t^{-2}$, an antiderivative for which is $F(t) = -t^{-1}$. Hence, the particle's position at time t is

$$s(t) = \int_1^t s'(u)du = F(u)\Big|_1^t = F(t) - F(1) = 1 - \frac{1}{t} < 1$$

for all $t \geq 1$. Thus, the particle will never pass $x = 1$, which implies it will never pass $x = 2$ either. 5.6.29

Problem 5.6.30 *Show that a particle, located at the origin at $t = 1$ and moving along the x-axis with velocity $v(t) = t^{-1/2}$, moves arbitrarily far from the origin after sufficient time has elapsed.*

SOLUTION. The particle's velocity is $v(t) = s'(t) = t^{-1/2}$, an antiderivative for which is $F(t) = 2t^{\frac{1}{2}}$. Hence, the particle's position at time t is

$$s(t) = \int_1^t s'(u)du = F(u)\Big|_1^t = F(t) - F(1) = 2\sqrt{t} - 1.$$

5.6.30

Problem 5.7.26

Evaluate the integral in the form $a\sin(u(x)) + C$ for an appropriate choice of $u(x)$ and constant a.

$$\int \cos x \cos(\sin x)\, dx$$

SOLUTION. Let $u = \sin x$. Then $du = \cos x\, dx$. Hence

$$\int \cos x \cos(\sin x)\, dx = \int \cos u\, du = \sin u + C = \sin(\sin x) + C$$

5.7.26

Problem 5.7.54 *Evaluate the indefinite integral.*

$$\int x^{-1/5} \sec(x^{4/5}) \tan(x^{4/5})\, dx$$

SOLUTION. Let $u = x^{4/5}$. Then $du = \frac{4}{5}x^{-1/5}\, dx$ so that $\frac{5}{4}du = x^{-1/5}\, dx$. Hence,

$$\int x^{-1/5} \sec(x^{4/5}) \tan(x^{4/5})\, dx = \frac{5}{4}\int \sec u \tan u\, du = \frac{5}{4}\sec u + C = \frac{5}{4}\sec(x^{4/5}) + C$$

5.7.54

Problem 5.7.64 *Hannah uses the substitution $u = \tan x$ and Akiva uses $u = \sec x$ to evaluate $\int \tan x \sec^2 x\, dx$. Show that they obtain different answers and explain the apparent contradiction.*

SOLUTION. With the substitution $u = \tan x$, Hannah finds that $du = \sec^2 x\, dx$. Thus

$$\int \tan x \sec^2 x\, dx = \int u \sec^2 x\, dx = \int u\, du = \frac{u^2}{2} + C_1 = \frac{\tan^2 x}{2} + C_1.$$

With the substitution $u = \sec x$, Akiva finds that $du = \tan x \sec x\, dx$. Thus

$$\int \tan x \sec^2 x\, dx = \int u(\tan x \sec x)\, dx = \int u\, du = \frac{u^2}{2} + C_2 = \frac{\sec^2 x}{2} + C_2.$$

One can check, by differentiating their answers, that Hannah and Akiva's answers are indeed both antiderivatives of $\tan x \sec^2 x$. To show that this is not a contradiction, recall that any two antiderivatives of a given function differ by a constant. Then, using the trigonometric identity $\tan^2 x + 1 = \sec^2 x$, we see that

$$\left(\frac{\tan^2 x}{2} + C_1\right) - \left(\frac{\sec^2 x}{2} + C_2\right) = \frac{\tan^2 x - \sec^2 x}{2} + C_1 - C_2 = \frac{1}{2} + C_1 - C_2.$$

So Hannah and Akiva's answers differ by a constant, as we should expect. 5.7.64

Problem 5.7.66

Some Choices Are Better Than Others *Evaluate*

$$\int \sin x \cos^2 x \, dx$$

in two ways.

First use $u = \sin x$ *to show that*

$$\int \sin x \cos^2 x \, dx = \int u\sqrt{1-u^2} \, du$$

and evaluate the integral on the right by a further substitution. Then show that $u = \cos x$ *is a better choice.*

SOLUTION. Consider the integral $\int \sin x \cos^2 x \, dx$. If we let $u = \sin x$, then $\cos x = \sqrt{1-u^2}$ and $du = \cos x \, dx$. Hence

$$\int \sin x \cos^2 x \, dx = \int u\sqrt{1-u^2} \, du.$$

Now let $w = 1 - u^2$. Then $dw = -2u \, du$ or $-\frac{1}{2} dw = u \, du$. Therefore,

$$\begin{aligned} \int u\sqrt{1-u^2} \, du &= -\frac{1}{2}\int w^{1/2} \, dw = -\frac{1}{2}\left(\frac{2}{3}w^{3/2}\right) + C \\ &= -\frac{1}{3}w^{3/2} + C = -\frac{1}{3}(1-u^2)^{3/2} + C \\ &= -\frac{1}{3}(1-\sin^2 x)^{3/2} + C = -\frac{1}{3}\cos^3 x + C. \end{aligned} \tag{1}$$

A better substitution choice is $u = \cos x$. Then $du = -\sin x \, dx$ or $-du = \sin x \, dx$. Hence

$$\int \sin x \cos^2 x \, dx = -\int u^2 \, du = -\frac{1}{3}u^3 + C = -\frac{1}{3}\cos^3 x + C.$$

5.7.66

Problem 5.7.83

Evaluate $\int_0^2 r\sqrt{5-\sqrt{4-r^2}} \, dr$.

SOLUTION. Choose $u = 5 - (4-r^2)^{1/2}$. Then

$$du = \frac{r \, dr}{(4-r^2)^{1/2}} = \frac{r \, dr}{5-u}$$

so that

$$(5-u) \, du = r \, dr$$

. When $x = 2$, $u = 5$, and when $x = 0$, $u = 3$, so our new upper limit will be 5 and our new lower limit will be 3 when we exchange the dx for the du.

Thus, the integral becomes

$$\int_0^2 r\sqrt{5-\sqrt{4-r^2}}\ dr = \int_3^5 (5-u)\sqrt{u}\ du = \int_3^5 (5u^{1/2} - u^{3/2})\ du$$

$$= (\frac{10}{3}u^{3/2} - \frac{2}{5}u^{5/2})\Big|_3^5 = (\frac{50}{3}\sqrt{5} - 10\sqrt{5}) - (10\sqrt{3} - \frac{18}{5}\sqrt{3}) = \frac{20}{3}\sqrt{5} - \frac{32}{5}\sqrt{3}.$$

5.7.83

Problem 5.7.95

Prove the formula $A = \pi ab$ *for the area of the ellipse with equation*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

SOLUTION. Here $a, b > 1$. By solving for y in terms of x in equation of the ellipse, we find that the area of the ellipse in the first quadrant is given by $\int_0^a \sqrt{b^2(1 - \frac{x^2}{a^2})}\ dx$.

Thus, the area of the ellipse is given by

$$4\int_0^a b\sqrt{1 - \frac{x^2}{a^2}}\ dx.$$

Recall that the equation of the unit circle is $u^2 + y^2 = 1$. By repeating the above work, we see that the area of the unit circle can be expressed by the integral

$$4\int_0^1 \sqrt{1-u^2}\ du = \pi.$$

Now, returning to our computation of the area of the ellipse, let us choose $u = \frac{x}{a}$, so $a\ du = dx$. Our integral becomes

$$4\int_0^a b\sqrt{1 - \frac{x^2}{a^2}}\ dx = 4b\int_0^1 \sqrt{1-u^2}a\ du = ab(4\int_0^1 \sqrt{1-u^2}\ du) = ab\pi.$$

5.7.95