Problem 6.1.19

Find the area of the shaded region.

SOLUTION. The equation of the line passing through $(\frac{\pi}{6}, \frac{\sqrt{3}}{2}$ is given by $y_1(x) = \frac{3\sqrt{3}}{\pi}x$, and the equation of the line passing through $(\frac{\pi}{3}, \frac{1}{2})$ is given by $y_2(x) = \frac{3}{2\pi}x$

The area of the region to the left of $x = \frac{\pi}{6}$ is

$$\int_0^{\frac{\pi}{6}} (y_1(x) - y_2(x))\, dx = \int_0^{\frac{\pi}{6}} (\frac{3\sqrt{3}}{\pi}x - \frac{3}{2\pi}x)\, dx = (\frac{3\sqrt{3}}{2\pi}x^2 - \frac{3}{4\pi}x^2)\Big|_0^{\frac{\pi}{6}}$$

$$= \frac{3\sqrt{3}}{2\pi}\frac{\pi^2}{36} - \frac{3}{4\pi}\frac{\pi^2}{36} = \frac{(2\sqrt{3}-1)\pi}{48}.$$

And the area of the region to the right of $x = \frac{\pi}{6}$ is

$$\int_{\pi/3}^{\pi/6} (\cos x - \frac{3}{2\pi}x)\, dx = (\sin x - \frac{3}{4\pi}x^2)\Big|_{\pi/3}^{\pi/6} = \frac{8\sqrt{3}-8-\pi}{16}.$$

Thus, the total area of the region is

$$\frac{(2\sqrt{3}-1)\pi}{48} + \frac{8\sqrt{3}-8-\pi}{16} = \frac{12\sqrt{3}-12+(\sqrt{3}-2)\pi}{24}.$$

6.1.19

Problem 6.1.23

Find the area of the region lying to the right of $x = y^2 + 4y - 22$ *and to the left of* $x = 3y + 8$.

SOLUTION. To figure out where the two curves intersect, we set $y^2 + 4y - 22 = 3y + 8$. This yields

$$0 = y^2 + y - 30 = (y-5)(y+6),$$

so the curves intersect at $y = 5$ and $y = -6$.

Thus, the area of the region is given by

$$\int_{-6}^{5} ((3y+8) - (y^2+4y-22))\, dy = \int_{-6}^{5} (-y^2 - y + 30)\, dy = (-\frac{1}{3}y^3 - \frac{1}{2}y^2 + 30y)\Big|_{-6}^{5} = \frac{1331}{6}$$

6.1.23

Problem 6.1.48

Sketch the region enclosed by the curves and compute its area as an integral along the x- or y-axis:

$y = \frac{\sin\sqrt{x}}{\sqrt{x}}, y = 0, \pi^2 \le x \le 9\pi^2$

SOLUTION. To compute the area enclosed by the curves, first we need to find where the two curves intersect, in the given interval. So we set $0 = \frac{\sin\sqrt{x}}{\sqrt{x}}$; and since $\sqrt{x}$ is nonzero on the interval $[\pi^2, 9\pi^2]$, this equation becomes

$$0 = \sin\sqrt{x}.$$

Since $0 = \sin\theta$ has solutions $\theta = n\pi$ for integers n, setting $\theta = \sqrt{x}$ shows that $\sin\sqrt{x}$ has solutions $x = n^2\pi^2$ for integers n. The only such solutions in $[\pi^2, 9\pi^2]$ are $x = \pi^2, 4\pi^2, 9\pi^2$

Now, we need to see which curve is above the other for the intervals $[\pi^2, 4\pi^2]$ and $[4\pi^2, 9\pi^2]$, respectively.

To do this, we choose a point x in the interior of each interval, and check if $\frac{\sin\sqrt{x}}{\sqrt{x}}$ is greater than or equal to, or less than or equal to, 0:

We choose $\frac{9}{4}\pi^2$ in $[\pi^2, 4\pi^2]$. Then

$$\frac{\sin\sqrt{\frac{9}{4}\pi^2}}{\sqrt{\frac{9}{4}\pi^2}} = \frac{\sin(\frac{3\pi}{2})}{\frac{3\pi}{2}} = \frac{-1}{\frac{3\pi}{2}} \le 0$$

Since these curves are continuous, and do not intersect in $(\pi^2, 4\pi^2)$, we know that $\frac{\sin\sqrt{x}}{\sqrt{x}} \le 0$ for all x in $[\pi^2, 4\pi^2]$

So the curve $y = \frac{\sin\sqrt{x}}{\sqrt{x}}$ is below the curve $y = 0$ on the interval $[\pi^2, 4\pi^2]$. Similar work will show that the curve $y = 0$ is below the curve $y = \frac{\sin\sqrt{x}}{\sqrt{x}}$ on the interval $[4\pi^2, 9\pi^2]$ (for example, you can plug in $x = \frac{25\pi^2}{4}$).

Thus, the area A enclosed by the two curves is given by

$$A = \int_{\pi^2}^{4\pi^2} (0 - \frac{\sin\sqrt{x}}{\sqrt{x}})\,dx + \int_{4\pi^2}^{9\pi^2} (\frac{\sin\sqrt{x}}{\sqrt{x}} - 0)\,dx$$

To evaluate this integral, choose $u = \sqrt{x} = x^{1/2}$. Then $du = \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2}\frac{1}{\sqrt{x}}\,dx$, so the integral becomes

$$A = -\int_{\pi^2}^{4\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx + \int_{4\pi^2}^{9\pi^2} (\frac{\sin\sqrt{x}}{\sqrt{x}})\,dx = -2\int_{\pi}^{2\pi} \sin u\,du + 2\int_{2\pi}^{3\pi^2} \sin u\,du$$
$$= 2\cos u\Big|_{\pi}^{2\pi} - 2\cos u\Big|_{2\pi}^{3\pi} = 4 + 4 = 8$$

6.1.48

Problem 6.1.63

Find the line $y = mx$ that divides the area under the curve $y = x(1-x)$ over $[0, 1]$ into two regions of equal area.

SOLUTION. First note that the area under the curve $y = x(1-x)$ is

$$\int_0^1 x(1-x)\,dx = \int_0^1 (x - x^2)\,dx = (\frac{1}{2}x^2 - \frac{1}{3}x^3)\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Now, the curves $y = mx$ and $y = x(1-x)$ intersect when $mx = x(1-x)$: at $x = 0$ or at $x = 1-m$.

[Note that if $m > 1$, then we'd have an intersection at a negative x value. In the context of our problem, this is nonsense: the line with a such a slope passes above the curve $y = x(1-x)$ on the interval $[0, 1]$, and therefore cannot divide the region into two parts. So a valid solution requires that $m \leq 1$.]

So the area A of the region enclosed by the curves $y = mx$ and $y = x(1-x)$ is given by

$$A = \int_0^{1-m} (x(1-x) - mx)\,dx = \int_0^{1-m} ((1-m)x - x^2)\,dx = (1-m)\frac{x^2}{2} - \frac{1}{3}x^3\Big|_0^{1-m} = \frac{1}{6}(1-m)^3$$

Setting $A = \frac{1}{2}\frac{1}{6} = \frac{1}{12}$, we get $\frac{1}{6}(1-m)^3 = \frac{1}{12}$, which requires

$$m = 1 - (\frac{1}{2})^{1/3}.$$

6.1.63

Problem 6.2.5

Find the volume of liquid needed to fill a sphere of radius R to height h.

SOLUTION. The horizontal cross section of the sphere at a height y above the sphere's lowest point is a circle. Its radius is $r(y) = \sqrt{R^2 - (R-y)^2}$, by the Pythagorean theorem.

So the area $A(y)$ of such a cross section at height y is given by $A(y) = \pi(r(y))^2 = \pi(R^2 - (R-y)^2)$.

To find the volume V of the region, we integrate these cross sectional areas for heights y ranging from 0 to h: that is,

$$V = \int_0^h \pi((R^2 - (R-y)^2))\,dy = \pi\int_0^h (2Ry - y^2)\,dy = \pi(Ry^2 - \frac{y^3}{3})\Big|_0^h = \pi(Rh^2 - \frac{h^3}{3}).$$

6.2.5

Problem 6.2.20

A plane inclined at an angle of 45 degrees passes through a diameter of the base of a cylinder of radius r. Find the volume of the region within the cylinder and below the plane.

SOLUTION. Place the origin at the center of the base of the cylinder, with the x-axis running perpendicularly to the diameter defined by the plane intersecting the cylinder. The point $x = r$ should lie directly below the highest point of the intersection of the plane and cylinder, as we've defined things here.

Now, for each x in $[0, r]$, the vertical cross section taken perpendicular to the x-axis is a rectangle of base $2\sqrt{r^2 - x^2}$ and height x.

Thus, the volume V of the region is

$$V \int_0^r 2x\sqrt{r^2 - x^2}\, dx$$

Choose $u = r^2 - x^2$. Then $du = -2x\, dx$, so our integral becomes

$$V \int_0^r 2x\sqrt{r^2 - x^2}\, dx = -\int_{r^2}^0 u^{1/2}\, du = -\frac{2}{3}u^{3/2}\Big|_{r^2}^0 = \frac{2}{3}r^3.$$

6.2.20

Problem 6.2.21

The solid S in Figure 25 is the intersection of two cylinders of radius r whose axes are perpendicular.

(a) *The horizontal cross section of each cylinder at distance y from the central axis is a rectangular strip. Find the strip's width.*

(b) *Find the area of the horizontal cross section of S at distance y.*

(c) *Find the volume of S as a function of r.*

SOLUTION. (a) Since our strip is a distance y above the central axis, and the radius of our cylinder is r, the Pythagorean theorem tells us that the width of the strip is $2\sqrt{r^2 - y^2}$.

(b) The horizontal cross section of S at distance y from the central axis is the intersection of the two rectangular strips at distance y from the central axis, from each cylinder. Since each strip has width $2\sqrt{r^2 - y^2}$, and the two rectangular strips intersect at right angles, this horizontal cross section is a square of width $2\sqrt{r^2 - y^2}$.

Thus, the area of the horizontal cross section of S at distance y is $(2\sqrt{r^2 - y^2})^2 = 4(r^2 - y^2)$.

(c) Since y ranges from $-r$ to r, the volume V of S is given by

$$V = \int_{-r}^r 4(r^2 - y^2)\, dy = 4(r^2y - \frac{1}{3}y^3)\Big|_{-r}^r = \frac{16}{3}r^3.$$

6.2.21

Problem 6.2.38

Let $v(r)$ be the velocity of blood in an arterial capillary of radius $R = 4 * 10^{-5}$ m. Use Poiseuille's Law (Example 6) with $k = 10^6$/(meter-seconds) to determine the velocity at the center of the capillary, and the flow rate (use correct units).

SOLUTION. Poiseuille's Law states that $v(r) = k(R^2 - r^2)$, but with R and r measured in cm and k measured in 1/(centimeter-seconds). So $R = 4 * 10^{-5}$ m = $4 * 10^{-5} * 10^2$ cm = $4 * 10^{-3}$ cm, and $k = 10^6/(\text{m-s}) = 10^6/(100\text{cm-s}) = 10^4/(\text{cm-s})$.

Now, the velocity at the center of the capillary is $v(0) = kR^2 = 10^4(4 * 10^{-3})^2 = 16 * 10^{-2}$ cm/s $= .0016$m/s.

The flow rate Q is given by

$$Q = 2\pi \int_0^R rv(r)\,dr = 2\pi \int_0^R rk(R^2 - r^2)\,dr = 2k\pi(\frac{R^2r^2}{2} - \frac{r^4}{4})\Big|_0^R$$

$$= 2\pi k \frac{R^4}{4} = 2\pi(10^4)(64 * 10^{-12}) = 1.28\pi * 10^{-6},$$

in cm^3/s.

This is the same as $1.28\pi * 10^{-12}$ m^3/s. 6.2.38

Problem 6.2.56 *An object with zero initial velocity accelerates at a constant rate of $10m/s^2$. Find the average velocity during the first 15 s.*

SOLUTION. An acceleration $a(t) = 10$ gives $v(t) = 10t + c$ for some constant c and zero initial velocity implies $c = 0$. Thus the average velocity is given by

$$\frac{1}{15-0}\int_0^{15} 10t\,dt = \frac{1}{3}t^2\Big|_0^{15} = 75m/s.$$

6.2.56

Problem 6.2.58 *What is the average area of the circles whose radii vary from 0 to R?*

SOLUTION. The average area is

$$\frac{1}{R-0}\int_0^R \pi r^2\,dr = \frac{\pi}{3R}r^3\Big|_0^R = \frac{1}{3}\pi R^2.$$

6.2.58

Problem 6.3.16 *Consider the functions $y = x^2$, $y = 2x + 3$.*

(a) Sketch the region enclosed by the curves.

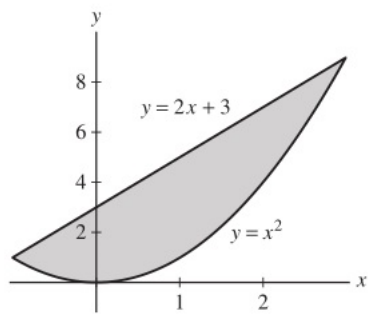
(b) Describe the cross section perpendicular to the x−axis located at x.

(c) Find the volume of the solid obtained by rotating the region about the x−axis.

SOLUTION. (a) Setting $x^2 = 2x + 3$ yields

$$0 = x^2 - 2x - 3 = (x-3)(x+1)$$

The two curves therefore intersect at $x = -1$ and $x = 3$. The region enclosed by the two curves is shown in the figure below.



(b) When the region is rotated about the x−axis, each cross section is a washer with outer radius $R = 2x + 3$ and inner radius $r = x^2$.

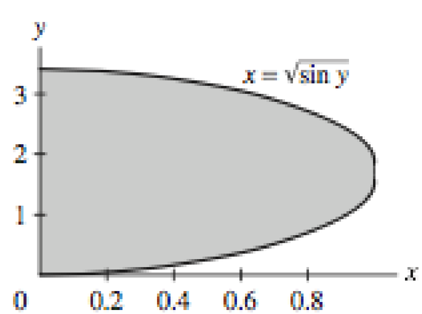
(c) The volume of the solid of revolution is

$$\pi \int_{-1}^{3} \left((2x+3)^2 - (x^2)^2 \right) dx = \pi \int_{-1}^{3} (4x^2 + 12x + 9 - x^4) dx$$
$$= \pi \left(\frac{4}{3}x^3 + 6x^2 + 9x - \frac{1}{5}x^5 \right) \Big|_{-1}^{3} = \frac{1088\pi}{15}$$

6.3.16

Problem 6.3.22 *Find the volume of the solid obtained by rotating the region enclosed by* $x = \sqrt{\sin(y)}$ *and* $x = 0$ *about the y-axis over the interval* $0 \le y \le \pi$

SOLUTION. Graphing the region yields the following:



Then rotating about the y-axis gives disks of radius $\sqrt{\sin(y)}$, so the volume is:

$$\int_0^{\pi} \pi R^2 dy = \pi \int_0^{\pi} \sin(y) dy = \pi(-\cos(y))\Big|_0^{\pi} = 2\pi$$

6.3.22

Problem 6.3.56 *The torus is obtained by rotating the circle* $(x-a)^2 + y^2 = b^2$ *around the y-axis (assume that* $a > b$*). Show that it has volume* $2\pi^2 ab^2$

SOLUTION. The image of the torus is as follows:

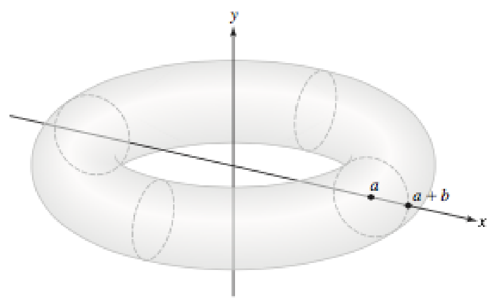


FIGURE 15 Torus obtained by rotating a circle about the y-axis.

Rotating the region enclosed by the circle $(x-a)^2 + y^2 = b^2$ about the Y—axis produces a torus whose cross sections are washers with outer radius $R = a + \sqrt{b^2 - y^2}$ and inner radius $r = a - \sqrt{b^2 - y^2}$.

The volume of the torus is then

$$\pi \int_{-b}^{b} \left(\left(a + \sqrt{b^2 - y^2}\right)^2 - \left(a - \sqrt{b^2 - y^2}\right)^2 \right) dy = 4a\pi \int_{-b}^{b} \sqrt{b^2 - y^2} dy.$$

Now, the remaining definite integral is one-half the area of a circle of radius B; therefore, the volume of the torus is

$$4a\pi . \frac{1}{2} \pi b^2 = 2\pi^2 ab^2$$

6.3.56

Problem 6.3.59 *A "bead" is formed by removing a cylinder of radius* r *from the center of a sphere of radius* R*. Find the volume of the bead with* $r = 1$ *and* $R = 2$*.*

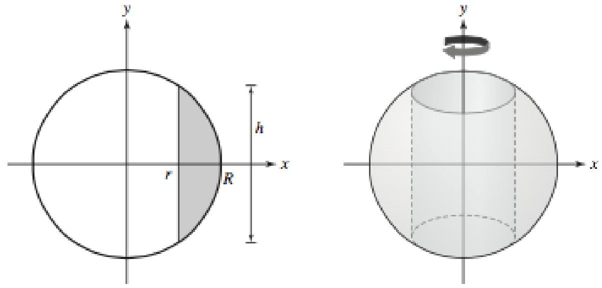


FIGURE 17 A bead is a sphere with a cylinder removed.

SOLUTION.

The equation of the outer circle is $x^2 + y^2 = 2^2 = 4$ and the inner cylinder intersects the sphere when $y = \pm\sqrt{3}$. Eachcross section of the bead is a washer with outer radius $\sqrt{4 - y^2}$ and inner radius 1, so the volume is given by

$$\pi \int_{-\sqrt{3}}^{\sqrt{3}} \left(\left(\sqrt{4 - y^2} \right)^2 - 1^2 \right) dy = \pi \int_{-\sqrt{3}}^{\sqrt{3}} (3 - y^2) dy = 4\pi\sqrt{3}$$

6.3.59