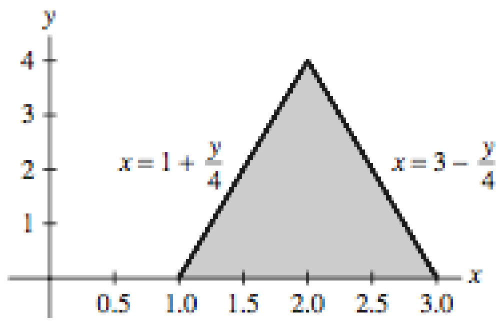
Problem 6.4.22 *Sketch the region enclosed by* $x = \frac{1}{4}y + 1$, $x = 3 - \frac{1}{4}y$, *and* $y = 0$. *Use the Shell Method to calculate the volume of rotation about the x-axis*

SOLUTION. The first equation is a line with positive slope and x-intercept $x = 1$, the second is a line with negative slope and x-intercept $x = 3$. These lines intersect at $\frac{1}{4}y + 1 = 3 - \frac{1}{4}y$ or $y = 4$. Then, our picture looks as follows:



Parallel to the x-axis we have shells with thickness dy, radius y, and height $3 - \frac{y}{4} - (1 + \frac{y}{4}) = 2 - \frac{y}{2}$. Thus, the volume of the solid is

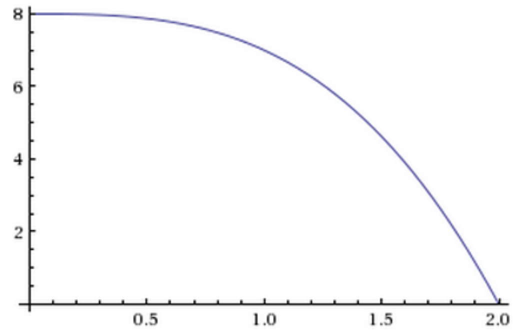
$$\int_0^4 2\pi(\text{radius})(\text{height of shell})\,dy = 2\pi\int_0^4 y(2 - \frac{y}{2})dy = 2\pi\int_0^4 2y - \frac{y^2}{2}\,dy$$
$$= 2\pi\,(y^2 - \frac{y^3}{6})\Big|_0^4 = 2\pi(16 - \frac{64}{6}) = \pi(32 - \frac{64}{3}) = \frac{32\pi}{3}$$

6.4.22

Problem 6.4.29 *Use both the Shell and Disk Methods to calculate the volume obtained by rotating the region under the graph of* $f(x) = 8 - x^3$ *from* $0 \le x \le 2$ *about*

- *the x-axis*
- *the y-axis*

SOLUTION. This region to be rotated is as follows:



- About the x-axis, the disk method gives disks of thickness dx and radius $y = 8-x^3$. Thus the volume is

$$\int_0^2 \pi(8-x^3)^2 dx = \pi \int_0^2 64 - 16x^3 + x^6 dx = \pi \left(64x - 4x^4 + \frac{x^7}{7}\right)\Big|_0^2 = \frac{576\pi}{7}$$

  Using the shell method, we get shells with thickness dy, radius y, and height $x = (8-y)^{\frac{1}{3}}$. Thus, the volume is

$$\int_0^8 2\pi y(8-y)^{\frac{1}{3}} dy$$

  Substituting $u = 8-y$ and $du = -dy$ we get

$$-2\pi \int_8^0 (8-u)u^{\frac{1}{3}} du = -2\pi \int_8^0 8u^{\frac{1}{3}} - u^{\frac{4}{3}} du = -2\pi \left(6u^{\frac{4}{3}} - \frac{3u^{\frac{7}{3}}}{7}\right)\Big|_8^0 = \frac{576\pi}{7}$$

- About the y-axis, disks have thickness dy and radius $x = (8-y)^{\frac{1}{3}}$ so we get volume

$$\int_0^8 \pi((8-y)^{\frac{1}{3}})^2 dy = \pi \int_0^8 (8-y)^{\frac{2}{3}} dy = \pi \left(-\frac{3}{5}(8-y)^{\frac{5}{3}}\right)\Big|_0^8 = \frac{96\pi}{5}$$

  Using the shell method, we get shells with thickness dx, radius x and height $y = 8-x^3$. Thus the volume is

$$\int_0^2 2\pi x(8-x^3) dx = 2\pi \int_0^2 8x - x^4 dx = 2\pi \left(4x^2 - \frac{x^5}{5}\right)\Big|_0^2 = \frac{96\pi}{5}$$

6.4.29

Problem 6.4.57 *Use the Shell Method to find the volume of the torus obtained by rotating the circle* $(x-a)^2 + y^2 = b^2$ *about the y-axis (assume* $a > b$*)*

SOLUTION. Our picture is again:

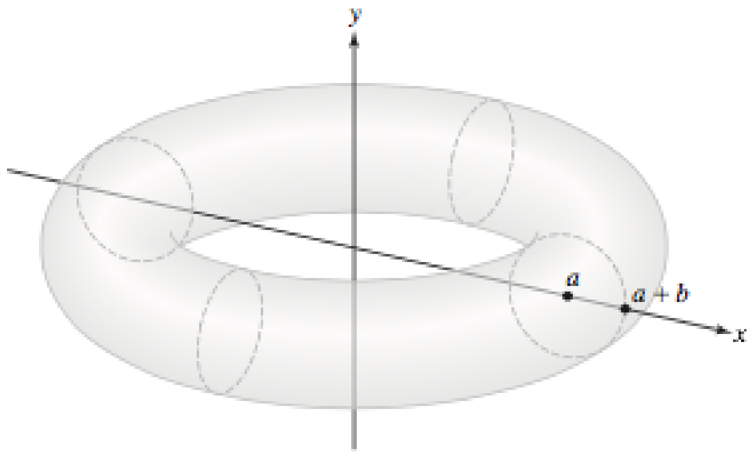


FIGURE 15 Torus obtained by rotating a circle about the y-axis.

Then we have cylinders with radius x going from $x = a - b$ to $x = a + b$. Each of these cylinders has height given by twice $y = \sqrt{b^2 - (x-a)^2}$ so we have volume:

$$\int_{a-b}^{a+b} 2\pi x(2\sqrt{b^2 - (x-a)^2})dx = 4\pi \int_{a-b}^{a+b} x\sqrt{b^2 - (x-a)^2}dx$$

Substituting $u = x - a$ and $du = dx$ we get

$$4\pi \int_{-b}^{b} (u+a)\sqrt{b^2 - u^2}du = 4\pi \int_{-b}^{b} u\sqrt{b^2 - u^2}du + 4\pi a \int_{-b}^{b} \sqrt{b^2 - u^2}du$$

Note that $u\sqrt{b^2 - u^2}$ is an odd function ($u\sqrt{b^2 - u^2} = -((-u)\sqrt{b^2 - (-u)^2}$) so it is 0 integrated about the symmetric interval [-b,b] (or use substitution $u = b\sin(\theta)$, $du = b\cos(\theta)d\theta$ to show this fact).

Then the volume is just $4\pi a \int_{-b}^{b} \sqrt{b^2 - u^2}du$. Since the integral here is again half of a circle of radius b, we have that the volume is $4\pi a(\frac{\pi b^2}{2}) = 2\pi^2 ab^2$.

6.4.57

Problem 6.4.60 *The surface area of a sphere of radius r is $4\pi r^2$. Use this to derive the formula for the volume V of a sphere of radius R in a new way*

- *Show that the volume of a thin spherical shell of inner radius r and thickness Δr is approximately* $4\pi r^2 \Delta r$
- *Approximate V by decomposing the sphere of radius R into N thin spherical shells of thickness* $\Delta r = \frac{R}{N}$
- *Show that the approximation is a Riemann sum that converges to an integral. Evaluate the integral*

SOLUTION. • The volume of such a spherical shell is approximately the surface area at the inner radius times the thickness, $4\pi r^2 \Delta r$. This is not exact because as we move Δr away from the inner radius r, the surface area of these spheres actually involves a slightly larger radius than r.

- Now we estimate V by decomposing the sphere into N spherical shells. The boundary points of these shells are given by $x_k = \frac{R}{N}k$, $k = 0, .., N$ (e.g. the first shell goes from $x_0 = 0$ to $x_1 = \frac{R}{N}$ and the last shell goes from $x_{N-1} = \frac{N-1}{N}R$ to $x_N = R$). Then, each shell has volume approximately equal to surface area at the outer radius times the thickness (e.g. the first shell has volume approximately $4\pi(x_1)^2\Delta r = 4\pi(\frac{R}{N})^2\frac{R}{N}$). This will give us an overestimation, for the same reason that part a described an underestimation.

  Thus, we can describe the approximate volume of the whole sphere as:

$$\sum_{k=1}^{N} 4\pi(x_k)^2\Delta r = \sum_{k=1}^{N} 4\pi(\frac{R}{N}k)^2\frac{R}{N} = 4\pi(\frac{R}{N})^3\sum_{k=1}^{N} k^2 = 4\pi R^3\frac{N(N+1)(2N+1)}{6N^3} = \frac{2}{3}\pi R^3\frac{(2N^3+3N^2+N)}{N^3}$$

- By definition, this is a Riemann sum partitioned by $0 = x_0 < x_1 \cdots < x_N = R$ and sampled at $x_1, \ldots, x_N$. Then, as we take the limit as N goes to infinity Δr becomes dr, x_k becomes r, and we now take r everywhere in $[0, R]$. Thus we obtain

$$\int_0^R 4\pi r^2\,dr = 4\pi\,(\frac{r^3}{3})\Big|_0^R = 4\pi\frac{R^3}{3}$$

6.4.60

Problem 6.5.19 *Calculate the work (in joules) required to pump all of the water out of the full tank below. Distances are in meters, the density of water is* $1000\frac{kg}{m^3}$

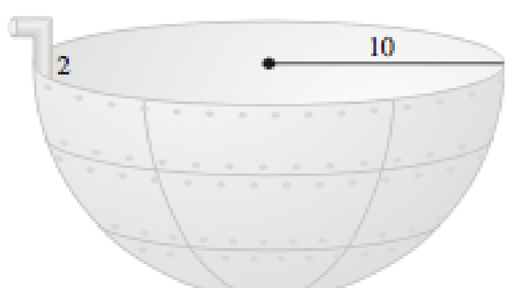


FIGURE 9

SOLUTION. The work against gravity to lift each layer of water of thickness dy is $F(\text{distance lifted}) = mg(distance) = area * dy * density * g * (distance)$. If we define the origin at the center of the hemisphere and let the positive y-axis point downward then a layer at depth y has radius $\sqrt{100 - y^2}$ by the Pythagorean theorem and thus area $\pi(100 - y^2)\ m^2$. This layer will have to be lifted $y + 2$ meters (because of the spout). Hence the work to lift this layer is $\pi(100 - y^2) * 1000 * 9.8 * (y + 2) * dy$.

So the total work to lift all layers is:

$$\int_{y=0}^{10} 9800\pi(100-y^2)(y+2)dy = 9800\pi \int_0^{10} 100y+200-y^3-2y^2 = 9800\pi\ (50y^2 + 200y - \frac{y^4}{4} - \frac{2y^3}{3})\Big|_0^{10}$$

$$= 9800\pi(\frac{11500}{3}) = \frac{112700000}{3}\pi \text{ joules}$$

6.5.19

Problem 6.5.20

Conical tank in Figure 10; water exits through the spout.

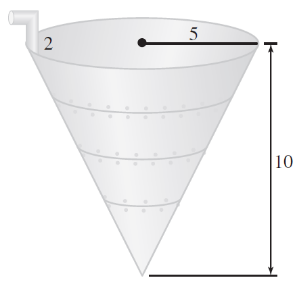


FIGURE 10

SOLUTION. Place the origin at the vertex of the inverted cone, and let the positive y-axis point upward. Consider a layer of water at a height of y meters.

From similar triangles, the area of the layer is

$$\pi\left(\frac{y}{2}\right)^2 m^2,$$

so the volume is

$$\pi\left(\frac{y}{2}\right)^2 \delta y m^3.$$

Thus the weight of one layer is

$$9800\pi\left(\frac{y}{2}\right)^2 \delta y N.$$

The layer must be lifted $12 - y$ meters, so the total work needed to empty the tank is

$$\int_0^{10} 9800\pi\left(\frac{y}{2}\right)^2 (12 - y)dy = \pi(3.675 \times 10^6)J \approx 1.155 \times 10^7 J.$$

6.5.20

Problem 6.5.27 *Calculate the work required to life a 10-m chain over the side of a building. Assume the chain has density* 8kg/m.

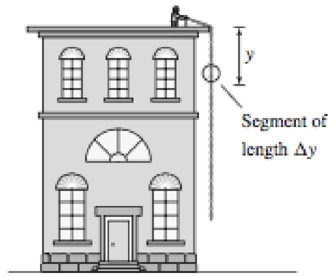


FIGURE 13 The small segment of the chain of length Δy located y meters from the top is lifted through a vertical distance y.

SOLUTION. We again consider the work against gravity to lift a length dy segment of chain (y meters down the rope). This is given by mg(distance). Here, the mass of the dy meter portion of chain is $8 * dy \frac{m(kg)}{m}$. This segment will be lifted y meters. Thus work to lift this segment is $8 * dy * 9.8 * y$. Then to lift all segments of the rope, starting from the top (0 meters down the rope) and ending at the bottom (10 meters down the rope), we have:

$$9.8 * 8 \int_{y=0}^{10} y\,dy = 78.4 \left(\frac{y^2}{2}\right)\Big|_0^{10} = 3920 \text{ joules}$$

6.5.27

Problem 6.5.35 *The gravitational force between two objects of mass m and M, separated by a distance r, has magnitude* $\frac{GMm}{r^2}$ *where* $G = 6.67 * 10^{-11} m^3 kg^{-1} s^{-1}$.

Show that if two objects of mass M and m are separated by a distance r_1*, then the work required to increase the separation to a distance* r_2 *is equal to* $W = GMm(r_1^{-1} - r_2^{-1})$

SOLUTION. Using the equation

$$W = \int_a^b F(x)dx$$

We obtain

$$W = \int_{r_1}^{r_2} \frac{GMm}{r^2} dr = GMm \int_{r_1}^{r_2} \frac{1}{r^2} dr = GMm \left(-\frac{1}{r}\right)\Big|_{r_1}^{r_2} = GMm(r_1^{-1} - r_2^{-1})$$

6.5.35

Problem 6.5.36 *Use the result of Exercise 35 to calculate the work required to place a 2000-kg satellite in an orbit 1200 km above the surface of the earth. Assume that the earth is a sphere of radius* $R_e = 6.37 * 10^6$ *m and mass* $M_e = 5.98 * 10^{24}$ *kg. Treat the satellite as a point mass.*

SOLUTION. The satellite moves from the surface of the earth to 1200*1000 m above the earth. This is from distance $r_1 = R_e$ m to $r_2 = R_e + 1200 * 1000 = R_e + 1200000$ m. Thus the work is

$$W = GMm(r_1^{-1} - r_2^{-1}) = GMm(\frac{1}{R_e} - \frac{1}{R_e + 1200000})$$

$$= (6.67 * 10^{-11})(5.98 * 10^{24})(2000)(\frac{1}{6.37 * 10^6} - \frac{1}{6.37 * 10^6 + 1200000}) \text{ joules}$$

6.5.36

Problem 7.1.18

Find the equation of the tangent line at the point indicated

$$y = e^{x^2}, \quad x_0 = 1$$

SOLUTION. Let $f(x) = e^{x^2}$. Then $f'(x) = 2xe^{x^2}$ and $f'(1) = 2e$. At $x_0 = 1$, $f(1) = e$, so the equation of the tangent line is $y = 2e(x - 1) + e = 2ex - e$. 7.1.18

Problem 7.1.44

Calculate the derivative indicated

$$\frac{d^2y}{dt}; \quad y = e^{-2t} \sin 3t$$

SOLUTION. Let $Y = e^{-2t} \sin 3t$. Then

$$\frac{dy}{dt} = e^{-2t}(3\cos 3t) - 2e^{-2t} \sin 3t = e^{-2t}(3\cos 3t - 2\sin 3t),$$

and

$$\frac{d^2y}{dt} = e^{-2t}(-9\sin 3t - 6\cos 3t) - 2e^{-2t}(3\cos 3t - 2\sin 3t) = e^{-2t}(-5\sin 3t - 12\cos 3t)$$

7.1.44

Problem 7.1.50 $f(x) = x^2e^x$

SOLUTION. Setting $f'(x) = (x^2 + 2x)e^x$ equal to zero and solving for x gives $(x^2 + 2x)e^x = 0$ which is true if and only if $x = 0$ or $x = -2$.

Now, $f''(x) = (x^2 + 4x + 2)e^x$. Because $f''(0) = 2 > 0, x = 0$ corresponds to a local minimum. On the other hand, $f''(-2) = (4 - 8 + 2)e^{-2} = -2/e^2 < 0$, so $x = -2$ corresponds to a local maximum.

7.1.50

Problem 7.1.81 $\int \frac{e^{2x} - e^{4x}}{e^x} dx$

SOLUTION.

$$\int \left(\frac{e^{2x} - e^{4x}}{e^x} \right) dx = \int (e^x - e^{3x}) dx = e^x - \frac{e^{3x}}{3} + C.$$

7.1.81

Problem 7.1.91

Wind engineers have found that wind speed v *(in m/s) at a given location follows a* **Rayleigh distribution** *of the type*

$$W(v) = \frac{1}{32} v e^{-v^2/64}$$

This means that the probability that v *lies between* a *and* b *is equal to the shaded area in the figure below*

1. *Show that the probability that* $v \in [0, b]$ *is* $1 - e^{-b^2/64}$.
2. *Calculate the probability that* $v \in [2, 5]$.

SOLUTION. 1. The probability that $v \in [0.b]$ is

$$\int_0^b \frac{1}{32} v e^{-v^2/64} dv.$$

Let $u = -v^2/64$. Then $du = -\frac{v}{32} dv$ and

$$\int_0^b \frac{1}{32} v e^{-v^2/64} dv = -\int_0^{-b^2/64} e^u du = 1 - e^{b^2/64}.$$

2. The probability that $v \in [2, 5]$ is the probability that $v \in [0, 5]$ minus the probability that $v \in [0, 2]$¿ Using part (a), it follows that the probability that $v \in [2, 5]$ is

$$(1 - e^{-25/64}) - (1 - e^{-4/64}) = e^{-1/16} - e^{-25/64} \approx 0.263.$$

7.1.91

Problem 7.1.94

Recall the following property of integrals: If $f(t) \geq g(t)$ *for all* $t \geq 0$, *then for all* $x \geq 0$,

$$\int_0^x f(t) dt \geq \int_0^x g(t) dt \qquad (1)$$

The inequality $e^t \geq 1$ *holds for* $t \geq 0$ *because* $e > 1$. *Use (1) to prove that*

$$e^x \geq 1 + x \text{ for } x \geq 0$$

Then prove, by successive integration, the following inequalities (for $x \geq 0$*):*

$$e^x \geq 1 + x + \frac{1}{2}x^2$$

$$e^x \geq 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

SOLUTION. Integrating both sides of the inequality $e^t \geq 1$ yields

$$\int_0^x e^t dt = e^x - 1 \geq x \text{ or } e^x \geq 1 + x.$$

Integrating both sides of this new inequality then gives

$$\int_0^x e^t \, dt = e^x - 1 \geq x + x^2/2 \text{ or } e^x \geq 1 + x + x^2/2.$$

Finally, integrating both sides again gives

$$\int_0^x e^t \, dt = e^x - 1 \geq x + x^2/2 + x^3/6 \text{ or } e^x \geq 1 + x + x^2/2 + x^3/6$$

as requested. 7.1.94