Problem 7.2.4

Show that $f(x) = \dfrac{x-2}{x+3}$ is invertible and find its inverse.

1. *What is the domain of f? The range of f^{-1}?*
2. *What is the domain of f^{-1}? The range of f?*

SOLUTION. We solve $y = f(x)$ for x as follows:

$$\begin{aligned} y =& \frac{x-2}{x+3} \\ yx + 3y =& x - 2 \\ yx - x =& -3y - 2 \\ x =& \frac{-3y-2}{y-1} = \frac{3y+2}{1-y}. \end{aligned}$$

Therefore,

$$f^{-1}(x) = \frac{3x+2}{1-x}.$$

1. Domain of $f(x) = \{x|x \neq -3\}$ = Range of $f^{-1}(x)$.
2. Domain of $f^{-1}(x) = \{x|x \neq 1\}$ = Range of $f(x)$.

7.2.4

Problem 7.2.18

Let n be a nonzero integer. Find a domain on which $f(c) = (1 - x^n)^{1/n}$ coincides with its inverse. Hint: The answer depends on whether n is even or odd.

SOLUTION. First note

$$f(f(x)) = \left(1 - \left((1-x^n)^{1/n}\right)^n\right)^{1/n} = (1 - (1 - x^n))^{1/n} = (x^n)^{1/n} = \begin{cases} x, & \text{if n is odd} \\ |x|, & \text{if n is even} \end{cases}$$

Let's first consider the case when $n > 0$. If n is even, then $f(x)$ is defined only when $1 - x^n \geq 0$. Hence, the domain of f is $[-1, 1]$. Now, in order for $f(f(x)) = x$ we need $|x| = x$, and so x must be nonnegative. Now, if we restrict the function f to the domain $[0, 1]$ we get that the range is $[0, 1]$, so $[0, 1]$ is the domain that we want. If n is odd, then $f(x)$ is defined for all real numbers, the range is also all real numbers, and $f(f(x)) = x$ for all x, so the domain we're looking for is $\mathbb{R}$.

Now, suppose $n < 0$. Then $-n > 0$, and

$$f(x) = \left(1 - \frac{1}{x^{-n}}\right)^{-1/-n} = \left(\frac{x^{-n}}{x^{-n} - 1}\right)^{1/-n}.$$

If n is even, then $f(x)$ is defined only when $x^{-n} - 1 > 0$. Hence, the domain of f is $|x| > 1$. Since we also want $f(f(x)) = |x|$ to be equal to x, we need to restrict f to the domain $x > 1$. In this case, the range is $y > 1$, so the domain $x > 1$ is the one we want. If n is odd, then $f(x)$ is defined for all real numbers except $x = 0$ (because then $1/x^{-n}$ is not defined) and $x = 1$ (because then $1/(x^{-n} - 1)$ is not defined), and so the range is all real numbers except $y = 1$, so the domain we're looking for is $\{x|x \neq 0,\ x \neq 1\}$.

7.2.18

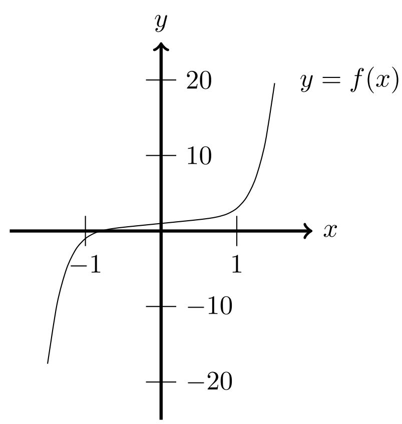
Problem 7.2.19

Let $f(x) = x^7 + x + 1$.

1. *Show that f^{-1} exists (but do not attempt to find it). Hint: Show that f is increasing.*
2. *What is the domain of f^{-1}?*
3. *Find $f^{-1}(3)$.*

SOLUTION.

1. The graph of $f(x) = x^7 + x + 1$ is shown below. From this graph, we see that $f(x)$ is a strictly increasing function; by Example 3, it is therefore one-to-one. Because f is one-to-one, by Theorem 3, f^{-1} exists.



2. The domain of $f^{-1}(x)$ is the range of $f(x) : (-\infty, \infty)$.
3. Note that $f(1) = 1^7 + 1 + 1 = 3$, therefore, $f^{-1}(3) = 1$.

7.2.19

Problem 7.2.24 *Let g be the inverse of $f(x) = x^3 + 1$. Find a formula for $g(x)$ and calculate $g'(x)$ in two ways: using Theorem 2 and then by direct calculation.*

SOLUTION. To find $g(x)$, we solve $y = x^3 + 1$ for x:

$$y - 1 = x^3$$

$$x = (y - 1)^{1/3}$$

Therefore, the inverse is $g(x) = (x-1)^{1/3}$. We have $f'(x) = 3x^2$. According to Theorem 2,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{3g(x)^2} = \frac{1}{3(x-1)^{2/3}} = \frac{1}{3}(x-1)^{-2/3}$$

This agrees with the answer we obtain by differentiating directly:

$$\frac{d}{dx}(x-1)^{1/3} = \frac{1}{3}(x-1)^{-2/3}.$$

7.2.24

Problem 7.2.32 *Find $g'(-\frac{1}{2})$, where g is the inverse of $f(x) = \frac{x^3}{x^2+1}$.*

Solution. Let $g(x)$ be the inverse of $f(x) = \frac{x^3}{x^2+1}$. Because

$$f(-1) = \frac{(-1)^3}{(-1)^2+1} = -\frac{1}{2},$$

it follows that $g(-\frac{1}{2}) = -1$. Moreover,

$$f'(x) = \frac{(x^2+1)(3x^2) - x^3(2x)}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2},$$

$$g'\left(-\frac{1}{2}\right) = \frac{1}{f'(g(-\frac{1}{2}))} = \frac{1}{f'(-1)} = 1.$$

7.2.32

Problem 7.3.6 $\log_2(8^{5/3})$

Solution.

$$\log_2(8^{5/3}) = \frac{5}{3}\log_2 2^3 = 5\log_2 2 = 5.$$

7.3.6

Problem 7.3.16 $8^{3\log_8(2)}$

Solution. Note that $3\log_8 2 = \log_8 2^3 = \log_8 8 = 1$. Thus, $8^{3\log_8 2} = 8^1 = 8$. 7.3.16

Problem 7.3.28

The **Gutenberg-Righter Law** *states that the number N of earthquakes per year worldwide of Richter magnitude at least M satisfies an approximate relation $\log_{10} N = a - M$ for some constant a. Find a, assuming that there is one earthquake of magnitude $M \geq 8$ per year. How many earthquakes of magnitude $M \geq 5$ occur per year?*

Solution. Substituting $N = 1$ and $M = 8$ into the Gutenberg-Richter law and solving for a yields

$$a = \log_{10}(1) + 8 = 0 + 8 = 8$$

The number N of earthquakes per year of Richter magnitude $M \geq 5$ satisfies

$$\log_{10}(N) = a - M = 8 - 5 = 3$$

so we conclude that $N = 10^3 = 1000$.

7.3.28

Problem 7.3.42

Find the derivative of $y = \ln((\ln x)^3)$.

SOLUTION. $\frac{d}{dx}\ln((\ln x)^3) = \frac{3(\ln x)^2}{x(\ln x)^3} = \frac{3}{x \ln x}$.

Alternately, because $\ln((\ln x)^3) = 3\ln(\ln x)$,

$$\frac{d}{dx}\ln((\ln x)^3) = 3\frac{d}{dx}\ln(\ln x) = 3\frac{1}{x \ln x}.$$

7.3.42

Problem 7.3.60 $f(x) = \ln(x^2),\ x = 4$

SOLUTION. Let $f(x) = \ln x^2 = 2\ln x$. Then $f(4) = 2\ln 4$. $f'(x) = 2/x$, so $f'(4) = 1/2$. Therefore, the equation of the tangent line is $y = (1/2)(x-4) + 2\ln 4$.

7.3.60

Problem 7.3.98 $\int \frac{dx}{x \ln x}$

SOLUTION. Let $u = \ln x$. Then $du = (1/x)dx$, and

$$\int \frac{dx}{x \ln x} = \int \frac{1}{u} du = \ln|u| + C = \ln|\ln x| + C.$$

7.3.98

Problem 7.3.100 $\int \frac{\ln(\ln x)}{x \ln x} dx$

SOLUTION. Let $u = \ln(\ln x)$. Then, $du = \frac{1}{x \ln x} dx$ and

$$\int \frac{\ln(\ln x)}{x \ln x} dx = \int u du = \frac{u^2}{2} + C = \frac{(\ln(\ln x))^2}{2} + C.$$

7.3.100

Problem 7.3.113

Find the minimum value of $f(x) = x^x$ *for* $x > 0$.

SOLUTION. Let $f(x) = x^x$. By Example 9 from the text, we know that $f'(x) = x^x(1 + \ln x)$. Thus, $x = \frac{1}{e}$ is the only critical point. Because $f'(x) < 0$ for $0 < x < \frac{1}{e}$ and $f'(x) > 0$ for $x > \frac{1}{e}$,

$$f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{1/e} \approx 0.692201$$

is the minimum value. 7.3.113

Problem 7.4.10

Find the function $y = f(t)$ that satisfies the differential equation $y' = 0.7y$ and the initial condition $y(0) = 10$.

SOLUTION. Given that $y' = -0.7y$ and $y(0) = 10$, then $f(t) = 10e^{-0.7t}$ 7.4.10

Problem 7.4.12

The half-life radium-226 is 1622 years. Find its decay constant.

SOLUTION. Half-life= $\frac{\ln 2}{k}$ so $k = \frac{\ln 2}{\textbf{half-life}} = \frac{\ln 2}{1622} = 4.27 \times 10^{-4}\text{years}^{-1}$. 7.4.12

Problem 7.4.30 *Assume that in a certain country, the rate at which jobs are created is proportional to the number of people who already have a job. If there are 15 million jobs at $t = 0$ and 15.1 million jobs 3 months later, how many jobs will there be after 2 years?*

SOLUTION. Let $J(t)$ denote the number of people, in millions, who have jobs at time t, in months. Because the rate at which jobs are created is proportional to the number of people who already have jobs, $J'(t) = kJ(t)$, for some constant k. Given that $J(0) = 15$, it then follows that $J(t) = 15e^{kt}$. To determine k, we use $J(3) = 15.1$; therefore,

$$k = \frac{1}{3} \ln\left(\frac{15.1}{15}\right) \approx 2.215 \times 10^{-3}.$$

Finally, after two years, there are

$$J(24) = 15e^{0.002215(24)} \approx 15.8 million$$

jobs.

7.4.30

Problem 7.4.32 *Verify that the half-life of a quantity that decays exponentially with decay constant k is equal to $\ln 2/k$.*

SOLUTION. Let $y = Ce^{-kt}$ be an exponential decay function. Let t be the half-life of the quantity y, that is the time t when $y = \frac{C}{2}$. Solving $\frac{C}{2} = Ce^{-kt}$ for t we get $-\ln 2 = -kt$, so $t = \ln 2/k$.

7.4.32