HOMEWORK SOLUTIONS

MATH 1910

Sections 7.7, 7.8

Fall 2016

Problem 7.7.10

SOLUTION. Because the quotient is not indeterminate at $x = 0$,

$$\frac{\cos x - \sin^2 x}{\sin x}|_{x=0} = \frac{1-0}{0} = \frac{1}{0}$$

L'Hôpital's Rule does not apply. $\boxed{7.7.10}$

Problem 7.7.12

SOLUTION. As $x \to \infty$, $x \sin \frac{1}{x}$ is of the form $\infty \cdot 0$, so L'Hôpital's Rule does not immediately apply. If we rewrite $x \sin \frac{1}{x}$ as $\frac{\sin \frac{1}{x}}{\frac{1}{x}}$, the rewritten expression if of the form $\frac{0}{0}$ as $x \to \infty$, so L'Hôpital's Rule now applies. We find

$$\lim_{x\to\infty} x \cdot \sin(\frac{1}{x}) = \lim_{x\to\infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\cos(\frac{1}{x})(\frac{-1}{x^2})}{\frac{-1}{x^2}} = \lim_{x\to\infty} \cos(\frac{1}{x}) = \cos 0 = 1.$$

$\boxed{7.7.12}$

Problem 7.7.45

SOLUTION.

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \lim_{x\to 0} \frac{\ln a \cdot a^x}{1} = \ln a.$$

$\boxed{7.7.45}$

Problem 7.7.48

SOLUTION.

$$\lim_{x\to 0^+} \ln(x^{\sin x}) = \lim_{x\to 0^+} \sin x(\ln x) = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\cos x(\sin x)^{-2}}$$

$$= \lim_{x\to 0^+} -\frac{\sin^2 x}{x \cos x} = \lim_{x\to 0^+} -\frac{2 \sin x \cos x}{-x \sin x + \cos x} = 0.$$

Hence, $\lim_{x\to 0^+} x^{\sin x} = \lim_{x\to 0^+} e^{\ln(x^{\sin x})} = e^0 = 1.$ $\boxed{7.7.48}$

Problem 7.7.54

Prove the following limit formula for e*:*

$$e = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$$

SOLUTION. Using the natural logarithm, we find

$$\ln\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right] = \lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}.$$

This last limit is indeterminate of the form $0/0$, so L'Hôpital's rule applies:

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = 1.$$

Thus,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^1 = e.$$

7.7.54

Problem 7.7.65

SOLUTION. (a) $1 \le 2+\sin x \le 3$, so

$$\frac{x}{x^2+1} \le \frac{x(2+\sin x)}{x^2+1} \le \frac{3x}{x^2+1}.$$

Since,

$$\lim_{x\to\infty}\frac{x}{x^2+1} = \lim_{x\to\infty}\frac{3x}{x^2+1} = 0,$$

it follows by the Squeez Theorem that

$$\lim_{x\to\infty}\frac{x(2+\sin x)}{x^2+1} = 0.$$

(b) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x(2+\sin x) \ge \lim_{x\to\infty} x = \infty$ and $\lim_{x\to\infty} g(x) = \lim_{x\to\infty}(x^2+1) = \infty$, but

$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \lim_{x\to\infty}\frac{x(\cos x)+(2+\sin x)}{2x}$$

does not exist since $\cos x$ oscillates. This does not violate L'Hôpital's Rule since the theorem clearly states

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

"provided the limit on the right exists."

7.7.65

Problem 7.7.66

SOLUTION. (a) Suppose $b \geq 1$. Then

$$H(b) = \lim_{x\to\infty} \frac{\ln(1+b^x)}{x} = \lim_{x\to\infty} \frac{b^x \ln b}{1+b^x} = \lim_{x\to\infty} \frac{b^x \ln b}{b^x} = \ln b.$$

(b) Now, suppose $0 < b < 1$. Then

$$H(b) = \lim_{x\to\infty} \frac{\ln(1+b^x)}{x} = \lim_{x\to\infty} \frac{b^x \ln b}{1+b^x} = \frac{0}{1} = 0.$$

7.7.66

Problem 7.8.62

SOLUTION. Let $x = \frac{2u}{5}$. Then

$$dx = \frac{2}{5} du,$$
$$4 - 25x^2 = 4(1-u^2)$$

and

$$\int_{-\frac{1}{5}}^{\frac{1}{5}} \frac{dx}{\sqrt{4-25x^2}} = \frac{2}{5} \int_{-\frac{1}{5}}^{\frac{1}{5}} \frac{1}{\sqrt{4(1-u^2)}} du$$
$$= \frac{1}{5} \sin^{-1} u|_{-\frac{1}{2}}^{\frac{1}{2}}$$
$$= \frac{1}{5}(\sin^{-1} \frac{1}{2} - \sin^{-1}(-\frac{1}{2}))$$
$$= \frac{\pi}{15}.$$

7.8.62

Problem 7.8.66

SOLUTION. Let $u = x^2$. Then $du = 2xdx$ and

$$\int \frac{xdx}{x^4+1} = \frac{1}{2} \int \frac{du}{u^2+1} = \frac{1}{2} \tan^{-1} x^2 + C.$$

7.8.66

Problem 7.8.79

SOLUTION. Let $u = \tan\theta$. Then $du = \sec^2\theta d\theta$ and

$$\int \sec^2\theta \tan^7\theta d\theta = \int u^7 du = \frac{1}{8}u^8 + C = \frac{1}{8}\tan^8\theta + C.$$

7.8.79

Problem 7.8.113

SOLUTION. Let $G(t) = \sqrt{1-t^2} + t\sin^{-1} t$. Then

$$\begin{aligned} G'(t) &= \frac{d}{dt}\sqrt{1-t^2} + \frac{d}{dt}(t\sin^{-1} t) \\ &= \frac{-t}{\sqrt{1-t^2}} + (t \cdot \frac{d}{dt}\sin^{-1} t + \sin^{-1} t) \\ &= \frac{-t}{\sqrt{1-t^2}} + (\frac{t}{\sqrt{1-t^2}} + \sin^{-1} t) \\ &= \sin^{-1} t. \end{aligned}$$

This proves the formula $\int \sin^{-1} t\,dt = \sqrt{1-t^2} + t\sin^{-1} t$. 7.8.113