HOMEWORK SOLUTIONS

MATH 1910

Sections 7.9, 8.1

Fall 2016

Problem 7.9.33 *Show that for any constants* M, k, *and* a, *the function*

$$y(t) = \frac{1}{2}M\left(1 + \tanh\left(\frac{k(t-a)}{2}\right)\right)$$

satisfies the ***logistic equation****:* $\dfrac{y'}{y} = k\left(1 - \dfrac{y}{M}\right)$.

SOLUTION. Let

$$y(t) = \frac{1}{2}M(1 + \tanh(\frac{k(t-a)}{2}))\ .$$

Then

$$1 - \frac{y(t)}{M} = \frac{1}{2}(1 - \tanh(\frac{k(t-a)}{2}))\ ,$$

and

$$ky(t)(1 - \frac{y(t)}{M}) = \frac{1}{4}Mk(1 - \tanh^2(\frac{k(t-a)}{2})) = \frac{1}{4}Mk^2(\frac{k(t-a)}{2})\ .$$

Finally,

$$y'(t) = \frac{1}{4}Mk^2(\frac{k(t-a)}{2}) = ky(t)(1 - \frac{y(t)}{M})\ .$$

7.9.33

Problem 7.9.54 *Solve the integral* $\displaystyle\int \frac{dx}{\sqrt{x^2-4}}$

SOLUTION.

$$\int \frac{dx}{\sqrt{x^2-4}} = \int \frac{d(x/2)}{\sqrt{(\frac{x}{2})^2 - 1}} = \cosh^{-1}(\frac{x}{2}) + C.$$

7.9.54

Problem 7.9.69

(a) *Show that* $y = \tanh t$ *satisfies the differential equation* $dy/dt = 1 - y^2$ *with initial condition* $y(0) = 0$.

(b) *Show that for arbitrary* A, B, *the function*

$$y = A\tanh(Bt)$$

satisfies

$$\frac{dy}{dt} = AB - \frac{B}{A}y^2, \qquad y(0) = 0$$

(c) *Let $\nu(t)$ be the velocity of a falling object of mass m. For large velocities, air resistance is proportional to the square velocity $\nu(t)^2$. If we choose coordinates so that $\nu(t) > 0$ for a falling object, then by Newton's Law of Motion, there is a constant $k > 0$ such that*

$$\frac{d\nu}{dt} = g - \frac{k}{m}\nu^2$$

Solve for $\nu(t)$ by applying the result of (b) with $A = \sqrt{gm/k}$ and $B = \sqrt{gk/m}$.

(d) *Calculate the terminal velocity $\lim_{t\to\infty} \nu(t)$.*

(e) *Find k if $m = 150$lb and the terminal velocity is 100mph.*

SOLUTION. (a) First, note that if we divide the identity $\cosh^2 t - \sinh^2 t = 1$ by $\cosh^2 t$, we obtain the identity $1 - \tanh^2 t = \operatorname{sech}^2 t$. Now, let $y = \tanh t$. Then,

$$\frac{dy}{dt} = \operatorname{sech}^2 t = 1 - \tanh^2 t = 1 - y^2.$$

Furthermore, $y(0) = \tanh 0 = 0$.

(b) Let $y = A\tanh(Bt)$. Then

$$\frac{dy}{dt} = AB\operatorname{sech}^2(Bt) = AB(1 - \tanh^2(Bt)) = AB\left(1 - \frac{y^2}{A^2}\right) = AB - \frac{By^2}{A}$$

Furthermore, $y(0) = A\tanh 0 = 0$.

(c) Matching the differential equation

$$\frac{d\nu}{dt} = g - \frac{k}{m}\nu^2$$

with the template

$$\frac{d\nu}{dt} = AB - \frac{B}{A}\nu^2$$

from part (b) yields

$$AB = g \quad \text{and} \quad \frac{B}{A} = \frac{k}{m}.$$

Solving for A and B gives

$$A = \sqrt{\frac{mg}{k}} \quad \text{and} \quad B = \sqrt{\frac{kg}{b}}.$$

Thus

$$\nu(t) = A\tanh(Bt) = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{b}}t\right).$$

(d) $\lim_{t\to\infty} \nu(t) = \sqrt{\frac{mg}{k}} \lim_{t\to\infty} \tanh\left(\sqrt{\frac{kg}{b}}t\right) = \sqrt{\frac{mg}{k}}$

(e) Substitute $m = 150\text{lb}$ and $g = 32\text{ft/sec}^2 = 78545.5\text{miles/hr}^2$ into the equation for the terminal velocity obtained in part (d) and then solve for k. This gives

$$k = \frac{150(78545.5)}{100^2} = 1178.18\text{lb/mile}$$

7.9.69

Problem 8.1.6 *Solve* $\int \tan^{-1} x dx$ *using integration by parts, with* $u = \tan^{-1} x$ *and* $dv = dx$.

SOLUTION. Using $u = \tan^{-1} x$ and $v' = 1$ gives us $u = \tan^{-1} x$, $v = x$

$\Longrightarrow u' = \frac{1}{x^2+1}$, $v' = 1$. Integration by Parts gives us

$$\int \tan^{-1} x dx = x \tan^{-1} x - \int (\frac{1}{x^2+1}) x dx.$$

For the integral on the right we'll use the substitution $w = x^2 + 1$, $dw = 2xdx$. Then we have

$$\int \tan^{-1} x dx = x \tan^{-1} x - \frac{1}{2}\int (\frac{1}{x^2+1}) 2x dx = x \tan^{-1} x - \frac{1}{2}\int \frac{dw}{w} =$$

$$= x \tan^{-1} x - \frac{1}{2} \ln|w| + C = x \tan^{-1} x - \frac{1}{2} \ln|x^2+1| + C.$$

8.1.6

Problem 8.1.13 *Solve the integral* $\int x^2 \sin x dx$.

SOLUTION. Let $u = x^2$ and $v' = \sin x$. Then we have $u = x^2 v = -\cos x$

$u' = 2x v' = \sin x$ Using Integration by Parts, we get

$$\int x^2 \sin x dx = x^2(-\cos x) - \int 2x(-\cos x) dx = -x^2 \cos x + 2\int x \cos x dx.$$

We must apply Integration by Parts again to evaluate $\int x \cos x dx$. Taking $u = x$ and $v' = \cos x$, we get

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + C.$$

Plugging this into the original equation gives us

$$\int x^2 \sin x dx = -x^2 \cos x + 2(x \sin x + \cos x) + C = -x^2 \cos x + 2x \sin x + 2\cos x + C.$$

8.1.13

Problem 8.1.20 *Solve the integral* $\int \frac{\ln x}{x^2} dx$.

SOLUTION. Let $u = \ln x$ and $v' = x^{-2}$. Then we have $u = \ln x v = -x^{-1}$

$u' = \frac{1}{x} v' = x^{-2}$ Using Integration by Parts, we get

$$\begin{aligned}\int \frac{\ln x}{x^2} dx &= -\frac{1}{x} \ln x - \int \frac{1}{x} (\frac{-1}{x}) dx = -\frac{1}{x} \ln x + \int x^{-2} dx \\ &= -\frac{1}{x} \ln x - \frac{1}{x} + C = -\frac{1}{x} (\ln x + 1) + C.\end{aligned}$$

8.1.20

Problem 8.1.38 *Solve the integral* $\int \frac{\ln(\ln x) dx}{x}$.

SOLUTION. Let $u = \ln(\ln x)$ and $dv = dx/x$. This gives $u' = \frac{(\ln x)'}{\ln x} = \frac{1}{x \ln x}$, and $v = \ln x$. Applying integration by parts formula, we have:

$$\int \frac{\ln(\ln x)}{x} dx = \ln x \cdot \ln(\ln x) - \int \ln x \cdot \frac{1}{x \cdot \ln x} dx = \ln x \cdot \ln(\ln x) - \ln x + c$$

8.1.38

Problem 8.1.49 *Solve the integral* $\int_1^2 x \ln x dx$.

SOLUTION. Let $u = \ln x$ and $dv = x dx$. This gives, $u' = 1/x dx$ and $v = x^2/2$. Using integration by parts formula we have:

$$\int_1^2 x \ln x dx = (\frac{x^2}{2} \ln x)_{1 \longrightarrow 2} - \int_1^2 \frac{x^2}{2} \cdot \frac{1}{x} dx = (\frac{x^2}{2} \ln x)_{1 \longrightarrow 2} - (\frac{x^2}{4})_{1 \longrightarrow 2} = 2 \ln 2 - 1 - (0 - \frac{1}{4}) = 2 \ln 2 - \frac{3}{4}$$

8.1.49

Problem 8.1.60

Derive the reduction formula

$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx.$$

SOLUTION. Let $u = x^n$ and $dv = e^x dx$. Then $du = n x^{x-1} dx$, $v = e^x$, and

$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx.$$

8.1.60

Problem 8.1.78

Find $f(x)$*, assuming that*

$$\int f(x) e^x dx = f(x) e^x - \int x^{-1} e^x dx$$

SOLUTION. We see that Integration by Parts was applied to $\int f(x)e^x dx$ with $u = f(x)$ and $dv = e^x dx$, and therefore $f'(x) = u' = x^{-1}$. Thus, $f(x) = \ln x + C$ for any constant C.

8.1.78

Problem 8.1.80

Find the area enclosed by $y = \ln x$ *and* $y = (\ln x)^2$.

SOLUTION. The two graphs intersect at $x = 1$ and $x = e$, and $\ln x$ is above $(\ln x)^2$, so the area is

$$\int_1^e \left(\ln x - (\ln x)^2\right) dx = \int_1^e \ln dx - \int_1^e (\ln x)^2 dx$$

Using integration by parts for the second integral, let $u = (\ln x)^2$, $dv = dx$; then $du = \frac{2\ln x}{x}$ and $v = x$, so that

$$\int_1^e (\ln x)^2 dx = \left(x(\ln x)^2\right)\Big|_1^e - 2\int_1^e \ln x dx = e - 2\int_1^e \ln x dx$$

Substituting this back into the original equation gives

$$\int_1^e \left(\ln x - (\ln x)^2\right) dx = 3\int_1^e \ln x dx - e$$

We use integration by parts to evaluate the remaining integral, with $u = \ln x$and $dv = dx$; then $du = \frac{1}{x} dx$ and $v = x$, so that

$$\int_1^e \ln x dx = x \ln x\Big|_1^e - int_1^e 1 dx = e - (e - 1) = 1$$

and thus, substituting back in, the value of the original integral is

$$\int_1^e \left(\ln x - (\ln x)^2\right) dx = 3\int_1^e \ln x dx - e = 3 - e$$

8.1.80

Problem 8.1.86

Define $P_n(x)$ *by*

$$\int x^n e^x dx = P_n(x)e^x + C$$

Use the reduction formula in Problem 60 to prove that $P_n(x) = x^n - nP_{n-1}(x)$. *Use this recursion relation to find* $P_n(x)$ *for* $n = 1, 2, 3, 4$. *Note that* $P_0(x) = 1$.

SOLUTION. From **8.1.60** we have

$$\int x^n e^x dx = x^n e^x - n\int x^{n-1} e^x dx = P_n(x)e^x + C \quad (1)$$

and also

$$\int x^{n-1} e^x dx = P_{n-1}(x)e^x + D \quad (2)$$

If we substitute the result of (2) into (1) and compare the coefficients in front of e^x we get:

$$P_n(x)e^x + C = x^n e^x - n(P_{n-1}(x)e^x + D) = e^x(x^n - nP_{n-1}(x)) - nD$$

which gives $P_n(x) = x^n - nP_{n-1}(x)$.

$$\begin{aligned}
P_1(x) &= x^1 - 1P_0(x) = x - 1 \\
P_2(x) &= x^2 - 2P_1(x) = x^2 - 2(x-1) = x^2 - 2x + 2 \\
P_3(x) &= x^3 - 3P_2(x) = x^3 - 3(x^2 - 2x + 2) = x^3 - 3x^2 + 6x - 6 \\
P_4(x) &= x^4 - 4P_3(x) = x^4 - 4(x^3 - 3x^2 + 6x - 6) = x^4 - 4x^3 + 12x^2 - 24x + 24
\end{aligned}$$

8.1.86

Problem 8.1.90

Set $I(a, b) = \int_0^1 x^a (1-x)^b dx$, *where* a, b *are whole numbers.*

(a) *Use substitution to show that* $I(a, b) = I(b, a)$.

(b) *Show that* $I(a, 0) = I(0, a) = \frac{1}{a+1}$.

(c) *Prove that for* $a \geq 1$ *and* $b \geq 0$,

$$I(a, b) = \frac{a}{b+1} I(a-1, b+1)$$

(d) *Use (b) and (c) to calculate* $I(1, 1)$ *and* $I(3, 2)$.

(e) *Show that* $I(a, b) = \frac{a!b!}{(a+b+1)!}$.

SOLUTION. (a) Let $u = 1 - x \implies du = -dx$ and the bounds of u go from 1 to 0.

$$I(a, b) = \int_0^1 x^a (1-x)^b dx = \int_1^0 (1-u)^a u^b (-du) = \int_0^1 (1-u)^a u^b du = I(b, a)$$

(b) For $b = 0$ from part (a) we get

$$I(a, 0) = I(b, 0) = \int_0^1 x^a dx = \left. \frac{x^{a+1}}{a+1} \right|_{0 \longrightarrow 1} = \frac{1}{a+1}$$

(c) Let us try to transform the RHS of what we want to prove, by using integration by parts, where $u = (1-x)^{b+1}$ and $dv = x^{a-1} dx$. This gives $du = -(b+1)x^b dx$ and $v = \frac{x^a}{a}$. Then we have:

$$\frac{a}{b+1} I(a-1, b+1) = \frac{a}{b+1} \left(\left. (1-x)^{b+1} \frac{x^a}{a} \right|_{0 \longrightarrow 1} + \int_0^1 \frac{b+1}{a} x^a (1-x)^b dx \right) = \int_0^1 x^a (1-x)^b dx = I(a, b)$$

(d)

$$I(1,1) = \frac{1}{2}I(1-1,1+1) = \frac{1}{2}I(0,2) = \frac{1}{2}I(2,0) = \frac{1}{2}\int_0^1 x^2 dx = \frac{1}{2}\left(\frac{x^3}{3}\right)_{0\longrightarrow 1} = \frac{1}{6}$$

$$I(3,2) = I(2,3) = \frac{2}{1+3}I(1,4) = \frac{2}{4}\frac{1}{5}I(0,5) = \frac{2}{20}I(5,0) = \frac{2}{20}\int_0^1 x^5 dx = \frac{2}{20}\frac{x^6}{6}_{0\longrightarrow 1} = \frac{1}{10}$$

(e) Apply result (c) a times and also apply result from part (b) once.

$$I(a,b) = \frac{a}{b+1}I(a{-}1,b{+}1) = \frac{a}{b+1}\cdot\frac{a-1}{b+2}I(a{-}2,b{+}2) = \frac{a}{b+1}\cdot\frac{a-1}{b+2}\frac{a-2}{b+3}I(a{-}3,b{+}3) = \cdots =$$

$$= \frac{a\cdot(a-1)\cdot(a-2)\cdots 2\cdot 1}{b\cdot(b+1)\cdot(b+2)\cdots(b+a-1)\cdot(b+a)}I(0,a+b) =$$

$$= \frac{a\cdot(a-1)\cdot(a-2)\cdots 2\cdot 1}{b\cdot(b+1)\cdot(b+2)\cdots(b+a-1)\cdot(b+a)}\cdot\frac{1}{b+a+1} = \frac{a!\cdot b!}{(a+b+1)!}$$

8.1.90

Problem 8.1.91

Let $I_n = \int x^n \cos(x^2)dx$ *and* $J_n = \int x^n \sin(x^2)dx$.

(a) Find a reduction formula that expresses I_n *in terms of* J_{n-2}. Hint: *Write* $X^n \cos(x^2)$ *and* $x^{n-1}(x\cos(x^2))$.

(b) Use the result of (a) to show that I_n *can be evaluated explicitly if* n *is odd.*

(c) Evaluate I_3.

SOLUTION. (a) Let $u = x^{n-1}$, $dv = x\cos(x^2) \Longrightarrow du = (n{-}1)x^{n-2}dx$ and $dv = \frac{1}{2}d(\sin(x^2)) \Longrightarrow v = \frac{1}{2}\sin(x^2)$. Applying integration by parts on I_n we get:

$$I_n = \int x^{n-1}(x\cos(x^2))dx = x^{n-1}\frac{\sin(x^2)}{2} - \frac{n-1}{2}\int x^{n-2}\sin(x^2)dx = \frac{x^{n-1}\cdot\sin(x^2)}{2} - \frac{n-1}{2}J_{n-2}$$

(b) If n is odd then $n-2, n-4, n-6, ..., 3, 1$ are all odd numbers, and recursively we can relate using result from part (a), I_n to $J_{n-2}, J_{n-4}, ..., J_1$ in the end. And we can easily compute J_1, which means that we can easily compute I_n for every n odd.

(c)

$$\begin{aligned} I_3 &= x^2\sin(x^2)/2 - J_1 = x^2\sin(x^2)/2 - \int x\sin(x^2)dx \\ &= x^2\sin(x^2)/2 + \int d(\cos(x^2))/2 \\ &= x^2\sin(x^2)/2 + \cos(x^2)/2 + C \end{aligned}$$

8.1.91