HOMEWORK SOLUTIONS

MATH 1910

Sections 8.2, 8.3, 8.5

Fall 2016

Problem 8.2.19

Evaluate using methods similar to those that apply to integral $\tan^m x \sec^n x$.

$$\int \cot^3 x dx$$

SOLUTION. Using the reduction formula for $\cot^m x$, we get

$$\int \cot^3 x dx = -\frac{1}{2}\cot^2 x - \int \cot x dx = -\frac{1}{2}\cot^2 x - \ln|\sin x| + C$$

8.2.19

Problem 8.2.27

$$\int \cos^3(\pi\theta)\sin^4(\pi\theta)d\theta$$

SOLUTION. Use the substitution $u = \pi\theta$, $du = \pi d\theta$, and the identity $\cos^2 u = 1 - \sin^2 u$ to write

$$\int \cos^3(\pi\theta)\sin^4(\pi\theta)d\theta = \frac{1}{\pi}\int \cos^3 u \sin^4 u du = \frac{1}{\pi}\int (1-\sin^2 u)\sin^4 u \cos u du$$

Now use the substitution $w = \sin u$, $dw = \cos u du$:

$$\int \cos^3(\pi\theta)\sin^4(\pi\theta)d\theta = \frac{1}{\pi}\int (1-w^2)w^4 dw = \frac{1}{5\pi}w^5 - \frac{1}{7\pi}w^7 + C = \frac{1}{5\pi}\sin^5(\pi\theta) - \frac{1}{7\pi}\sin^7(\pi\theta) + C$$

8.2.27

Problem 8.2.65

Find the volume of the solid obtained by revolving $y = \sin x$ *for* $0 \le x \le \pi$ *about the x-axis.*

SOLUTION.

$$\int_0^\pi \pi(\sin x)^2 dx = \pi(\frac{x}{2} - \frac{\sin 2x}{4})\Big|_0^\pi = \frac{\pi^2}{2}$$

8.2.65

Problem 8.2.68

Evaluate the integral $J = \int \sin^m x \cos^n x\, dx$ *where* m *and* n *are both even using the identities* $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ *and* $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$*:* $J = \int \cos^4 x\, dx$

SOLUTION. Using the second identity,

$$J = \frac{1}{4}\int (1+\cos 2x)^2\, dx = \frac{1}{4}\int (1+2\cos 2x+\cos^2 2x)\, dx = \frac{1}{4}\int 1\, dx + \frac{1}{4}\int 2\cos 2x\, dx + \frac{1}{4}\int \frac{1}{2}(1+\cos 4x)\, dx$$

Using the substitution, we see

$$J = \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + C = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

8.2.68

Problem 8.3.3

$$I = \int \frac{dx}{\sqrt{4x^2+9}}$$

(a) *Show that the substitution* $x = \frac{3}{2}\tan\theta$ *transforms* I *into* $\frac{1}{2}\int \sec\theta d\theta$

(b) *Evaluate* I *in terms of* θ.

(c) *Express* I *in terms of* x.

SOLUTION. (a) if $x = \frac{3}{2}\tan\theta$. Then $dx = \frac{3}{2}\sec^2\theta d\theta$ and

$$\sqrt{4x^2+9} = \sqrt{9\tan^2\theta + 9} = 3\sqrt{\sec^2\theta} = 3\sec\theta$$

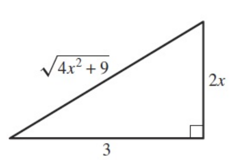
Thus,

$$I = \int \frac{dx}{\sqrt{4x^2+9}} = \int \frac{\frac{3}{2}\sec^2\theta d\theta}{3\sec\theta} = \frac{1}{2}\int \sec\theta d\theta$$

(b)

$$I = \frac{1}{2}\int \sec\theta d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

(c) Since $x = \frac{3}{2}\tan\theta$, we construct a right angle triangle with $\tan\theta = \frac{2x}{3}$



From this we see that $\sec\theta = \frac{1}{3}\sqrt{4x^2+9}$, and so

$$I = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \frac{1}{2}\ln|\frac{1}{3}\sqrt{4x^2+9} + \frac{2x}{3}| + C$$

$$= \frac{1}{2}\ln|\frac{\sqrt{4x^2+9}+2x}{3}| + C = \frac{1}{2}\ln|\sqrt{4x^2+9}+2x| - \frac{1}{2}\ln 3 + C = \frac{1}{2}\ln|\sqrt{4x^2+9}+2x| + C$$

8.3.3

Problem 8.3.15

Evaluate using trigonometric substitution. Refer to the table of trigonometric integrals as necessary.

$$\int \frac{x^2 dx}{\sqrt{9-x^2}}$$

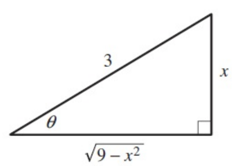
SOLUTION. Let $x = 3\sin\theta$. Then $dx = 3\cos\theta d\theta$

$$9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta$$

and

$$I = \int \frac{x^2 dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta(3\cos\theta d\theta)}{3\cos\theta} = 9\int \sin^2\theta d\theta = 9(\frac{\theta}{2} - \frac{1}{2}\sin\theta\cos\theta) + C$$

Since $x = 3\sin\theta$, we construct a right angle triangle with $\sin\theta = \frac{x}{3}$



From this we see that $\cos\theta = \frac{\sqrt{9-x^2}}{3}$, and so

$$I = \frac{9}{2}\sin^{-1}(\frac{x}{3}) - \frac{9}{2}(\frac{x}{3})(\frac{\sqrt{9-x^2}}{3}) + C = \frac{9}{2}\sin^{-1}(\frac{x}{3}) - \frac{1}{2}x\sqrt{9-x^2} + C$$

8.3.15

Problem 8.3.37

$$\int \frac{dx}{\sqrt{x^2+4x+13}}$$

SOLUTION. First complete the squarre:

$$x^2 + 4x + 13 = x^2 + 4x + 4 + 9 = (x+2)^2 + 9$$

Let $u = x + 2$. Then $dx = du$, and

$$I = \int \frac{dx}{\sqrt{x^2 + 4x + 13}} = \int \frac{dx}{\sqrt{(x+2)^2 + 9}} = \int \frac{du}{\sqrt{u^2 + 9}}$$

Now let $u = 3tan\theta$. Then $du = 3sec^2\theta d\theta$,

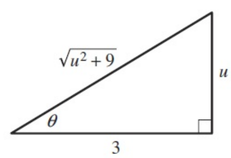
$$u^2 + 9 = 9tan^2\theta + 9 = 9(tan^2\theta + 1) = 9sec^2\theta$$

,

and

$$I = \int \frac{3sec^2\theta d\theta}{3sec\theta} = \int sec\theta d\theta = \ln|sec\theta + tan\theta| + C$$

Since $u = 3tan\theta$, we construct the following right triangle:



From this we see that $\sec\theta = \frac{\sqrt{u^2+9}}{3}$. Thus

$$I = \ln|\frac{\sqrt{u^2+9}}{3} + \frac{u}{3}| + C_1 = \ln|\sqrt{u^2+9} + u| + (\ln\frac{1}{3} + C_1)$$
$$= \ln|\sqrt{(x+2)^2 + 9} + x + 2| + C$$

8.3.37

Problem 8.3.46

$$\int x^2 \ln(x^2 + 1) dx$$

SOLUTION. Start by using integration by parts with $u = \ln(x^2 + 1)$, $dv = x^2 dx$, then $du = \frac{2x}{x^2+1} dx$ and $v = \frac{1}{3}x^3$, so that

$$I = \int x^2 \ln(x^2 + 1) dx = \frac{1}{3}x^3 \ln(x^2 + 1) - \frac{2}{3}\int \frac{x^4}{x^2 + 1} dx$$

By long division,

$$\frac{x^4}{x^2+1} = x^2 - 1 + \frac{1}{x^2+1}$$

, so

$$\int \frac{x^4}{x^2+1}dx = \int (x^2 - 1 + \frac{1}{x^2+1})dx = \frac{1}{3}x^3 - x + \tan^{-1}x + C$$

Therefore,

$$I = \frac{1}{3}x^3\ln(x^2+1) - \frac{2}{3}(\frac{1}{3}x^3 - x + \tan^{-1}x) + C$$

8.3.46

Problem 8.3.48

Find the volume of the solid obtained by revolving the graph of $y = x\sqrt{1-x^2}$ over $[0,1]$ about the y-axis.

SOLUTION. Using the method of cylindrical shells, the volume is given by

$$V = 2\pi\int_0^1 x(x\sqrt{1-x^2})\ dx = 2\pi\int_0^1 x^2\sqrt{1-x^2}\ dx$$

To evaluate this, let $x = \sin\theta$, $dx = \cos\theta d\theta$ so

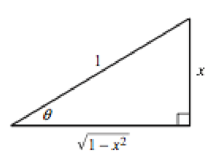
$$\int x^2\sqrt{1-x^2}\ dx = \int \sin^2\theta\sqrt{1-\sin^2\theta}\cos\theta d\theta = \int \sin^2\theta\cos^2 d\theta = \int \cos^2\theta - \cos^4\theta\ d\theta$$

Using the reduction formula for $\int\cos^4\theta d\theta$,

$$\int \cos^2\theta - \cos^4\theta\ d\theta = \int\cos^2\theta d\theta - \left[\frac{\cos^3\theta\sin\theta}{4} + \frac{3}{4}\int\cos^2\theta\ d\theta\right] = -\frac{1}{4}\cos^3\theta\sin\theta + \frac{1}{4}\int\cos^2\theta d\theta$$

$$= -\frac{1}{4}\cos^3\theta\sin\theta + \frac{1}{4}\left[\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right] + C = -\frac{1}{4}\cos^3\theta\sin\theta + \frac{1}{8}\theta + \frac{1}{8}\sin\theta\cos\theta + C$$

Since $\sin\theta = x$, we use the following triangle to see that $\cos\theta = \sqrt{1-x^2}$



Thus,

$$\int \cos^2\theta - \cos^4\theta\ d\theta = -\frac{1}{4}(1-x^2)^{3/2}x + \frac{1}{8}\arcsin x + \frac{1}{8}x\sqrt{1-x^2} + C$$

Using this, we compute the volume:

$$V = 2\pi\left(-\frac{1}{4}x(1-x^2)^{3/2} + \frac{1}{8}\arcsin x + \frac{1}{8}x\sqrt{1-x^2}\right)\Bigg|_0^1 = 2\pi\left[\left(0+\frac{\pi}{16}+0\right)-(0)\right] = \frac{\pi^2}{8}$$

8.3.48

Problem 8.3.49

Find the volume of the solid obtained by revolving the region between the graph of $y^2 - x^2 = 1$ and the line $y = 2$ about the line $y = 2$.

SOLUTION. First solve the equation $y^2 - x^2 = 1$ for y:

$$y = \pm\sqrt{x^2+1}$$

The region in question is bounded in part by the top half of this hyperbola, which is the equation.

$$y = \sqrt{x^2+1}$$

The limits of integration are obtained by finding the points of intersection of this equation with $y = 2$:

$$2 = \sqrt{x^2+1} \to x = \pm\sqrt{3}$$

The radius of each disk is given by $2 - \sqrt{x^2+1}$; the volume is therefore given by

$$V = \int_{-\sqrt{3}}^{\sqrt{3}} \pi r^2 dx = 2\pi\int_0^{\sqrt{3}} (2-\sqrt{x^2+1})^2 dx =$$

$$2\pi\int_0^{\sqrt{3}} [4 - 4\sqrt{x^2+1} + (x^2+1)]dx$$

$$= 8\pi\int_0^{\sqrt{3}} dx - 8\pi\int_0^{\sqrt{3}} \sqrt{x^2+1}dx + 2\pi\int_0^{\sqrt{3}} (x^2+1)dx$$

To evaluate the integral $\int \sqrt{x^2+1}dx$, let $x = tan\theta$. Then $dx = sec^2\theta d\theta$, $x^2+1 = sec^2\theta$

$$\int \sqrt{x^2+1}dx = \int sec^3\theta d\theta = \frac{1}{2}tan\theta sec\theta + \frac{1}{2}\int sec\theta d\theta$$

$$= \frac{1}{2}tan\theta sec\theta + \frac{1}{2}ln|sec\theta + tan\theta| + C$$

$$= \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}ln|\sqrt{x^2+1} + x| + C$$

Now we can compute the volume:

$$V = \left[8\pi x - 8\pi(\frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln|\sqrt{x^2+1}+x|) + \frac{2}{3}\pi x^3 + 2\pi x\right]\Big|_0^{\sqrt{3}} = 4\pi[\sqrt{3} - \ln|2+\sqrt{3}|]$$

8.3.49

Problem 8.5.4 *Find the constants in the partial fraction decomposition*

$$\frac{2x+4}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$$

SOLUTION. Clearing denominators gives:

$$2x+4 = A(x^2+4) + (Bx+C)(x-2)$$

Setting $x = 2$ then yields: $A = 1$ To find B and C, expand the right side, gather like terms, and use the method of undetermined coefficients:

$$2x+4 = (B+1)x^2 + (-2B+C)x + (4-2C)$$

Equating x^2 coefficients we find $B = -1$. While equating constants yields $C = 0$.

8.5.4

Problem 8.5.15 *Evaluate the integral.*

$$\int \frac{(x^2+3x-44)dx}{(x+3)(x+5)(3x-2)}dx$$

SOLUTION. The partial fraction decomposition has the form:

$$\frac{(x^2+3x-44)}{(x+3)(x+5)(3x-2)} = \frac{A}{x+3} + \frac{B}{x+5} + \frac{C}{3x-2}$$

Clearing denominators gives us

$$x^2+3x-44 = A(x+5)(3x-2) + B(x+3)(3x-2) + C(x+3)(x+5)$$

. Setting $x = -3$ then yields $A = 2$, while setting $x = -5$ yields $B = -1$ and setting $x = \frac{2}{3}$ yields $C = -2$.

The result is:

$$\frac{(x^2+3x-44)}{(x+3)(x+5)(3x-2)} = \frac{2}{x+3} + \frac{-1}{x+5} + \frac{-2}{3x-2}$$

Thus

$$\int \frac{(x^2+3x-44)dx}{(x+3)(x+5)(3x-2)}dx = 2\int \frac{dx}{x+3} - \int \frac{dx}{x+5} - 2\int \frac{dx}{3x-2}$$

$$= 2\ln|x+3| - \ln|x+5| - \frac{2}{3}\ln|3x-2| + C$$

8.5.15

Problem 8.5.52 *Evaluate* $\int \frac{dx}{x^{1/2}-x^{1/3}}$. *Hint: Use the substitution* $u = x^{1/6}$.

SOLUTION. By long division and substitution $u = x^{1/6}$, $du = \frac{1}{6}x^{-5/6}dx \to 6x^{5/6}du = dx \to 6u^5 du = dx$

$$\int \frac{dx}{x^{1/2}-x^{1/3}} = \int \frac{6u^5}{u^3-u^2}\,du = 6\int \frac{u^3}{u-1}\,du = 6\int \left(u^2+u+1+\frac{1}{u-1}\right)\,du =$$

$$6\left(\frac{1}{3}u^3+\frac{1}{2}u^2+u+\ln|u-1|\right)+C = 2u^3+3u^2+6u+6\ln|u-1|+C = 2x^{1/2}+3x^{1/3}+6x^{1/6}+6\ln|x^{1/6}-1|+C$$

8.5.52

Problem 8.5.58 *Use the substitution of Exercise 57 to evaluate* $\int \frac{d\theta}{\cos\theta+\sin\theta}$

SOLUTION. Using the substitution $\theta = 2\tan^{-1}t$, we get

$$\int \frac{d\theta}{\cos\theta+\sin\theta} = \int \frac{\frac{2dt}{(1+t^2)}}{\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}} = \int \frac{2dt}{1-t^2+2t} = -2\int \frac{dt}{t^2-2t-1}$$

The partial fraction decomposition has the form

$$\frac{-2}{t^2-2t-1} = \frac{A}{t-1-\sqrt{2}} + \frac{B}{t-1+\sqrt{2}}$$

Clearing denominators gives us

$$-2 = A(t-1+\sqrt{2}) + B(t-1-\sqrt{2})$$

Setting $t = 1+\sqrt{2}$ then yields $A = -\frac{1}{\sqrt{2}}$, while setting $t = 1-\sqrt{2}$ yields $B = \frac{1}{\sqrt{2}}$. Thus,

$$\int \frac{d\theta}{\cos\theta+\sin\theta} = \frac{1}{\sqrt{2}}\int \frac{dt}{t-1+\sqrt{2}} - \frac{1}{\sqrt{2}}\int \frac{dt}{t-1-\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\ln|t-1+\sqrt{2}| - \frac{1}{\sqrt{2}}\ln|t-1-\sqrt{2}| + C$$

$$= \frac{1}{\sqrt{2}}\ln\left|\frac{\tan(\frac{\theta}{2})-1+\sqrt{2}}{\tan(\frac{\theta}{2})-1-\sqrt{2}}\right| + C$$

8.5.58

Problem 8.5.62

Suppose that $Q(x) = (x-a_1)(x-a_2)...(x-a_n)$*, where the roots* a_j *are all distinct. Let* $\frac{P}{Q}$ *be a proper rational function so that*

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + ... + \frac{A_n}{x-a_n}$$

(a) Show that $A_j = \frac{P(a_j)}{Q'(a_j)}$ *for* $j = 1, ..., n$.

(b) Use this result to find the partial fraction decomposition for $P(x) = 2x^2 - 1$, $Q(x) = x^3 - 4x^2 + x + 6 = (x+1)(x-2)(x-3)$.

SOLUTION. To differentiate $Q(x)$, first take the logarithm of both sides, and then differentiate:

$$\ln(Q(x)) = \ln[(x-a_1)(x-a_2)...(x-a_n)] = \ln(x-a_1) + \ln(x-a_2) + ... + \ln(x-a_n)$$

$$\frac{d}{dx}\ln(Q(x)) = \frac{Q'(x)}{Q(x)} = \frac{1}{x-a_1} + \frac{1}{x-a_2} + ... + \frac{1}{x-a_n}$$

Multiplying both sides by $Q(x)$ gives us

$$Q'(x) = Q(x)[\frac{1}{x-a_1} + ... + \frac{1}{x-a_n}]$$

$$= (x-a_2)(x-a_3)...(x-a_n) + (x-a_1)(x-a_3)...(x-a_n) + ... + (x-a_1)(x-a_2)...(x-a_{n-1})$$

In other words, the ith product in the formula for $Q'(x)$ has the $(x-a_i)$ factor removed. This means that

$$Q'(a_j) = (a_j - a_1)...(a_j - a_{j-1})(a_j - a_{j+1})...(a_j - a_n)$$

.

Now clear the denominators in the expression for $\frac{P(x)}{Q(x)}$:

$$P(x) = \frac{A_1 Q(x)}{x-a_1} + ... + \frac{A_n Q(x)}{x-a_n}$$

$$= A_1(x-a_2)...(x-a_n) + (x-a_1)A_2(x-a_3)...(x-a_n) + ... + (x-a_1)(x-a_2)...(x-a_{n-1})A_n$$

Setting $x = a_j$, we get

$$P(a_j) = (a_j - a_1)(a_j - a_2)...(a_j - a_{j-1})A_j(a_j - a_{j+1})...(a_j - a_n))$$

So that

$$A_j = \frac{P(a_j)}{(a_j - a_1)...(a_j - a_{j-1})(a_j - a_{j+1})...(a_j - a_n)} = \frac{P(a_j)}{Q'(a_j)}$$

(b) Let $P(x) = 2x^2 - 1$ and $Q(x) = (x+1)(x-2)(x-3)$, so that $Q'(x) = 3x^2 - 8x + 1$. Then $a_1 = -1$, $a_2 = 2$, and $a_3 = 3$, so that 8.5.62

$$A_1 = \frac{P(-1)}{Q'(-1)} = \frac{1}{12}$$

;

$$A_2 = \frac{P(2)}{Q'(2)} = \frac{-7}{3}$$

;

$$A_3 = \frac{P(3)}{Q'(3)} = \frac{17}{4}$$

;

Thus

$$\frac{P(x)}{Q(x)} = \frac{1}{12(x+1)} - \frac{7}{3(x-2)} + \frac{17}{4(x-3)}$$