

5.4 Euler Equations

In this section we consider equations of the form $P(x)y'' + Q(x)y' + R(x)y = 0 \star$

in the neighborhood of a singular point x_0 .
[i.e. a point x_0 , s.t. $p(x_0) = 0$].

to simplify \star further, we look at

$$L[y] = x^2 y'' + \alpha x y' + \beta y = 0 \text{ for } \alpha, \beta \in \mathbb{R}.$$

Try to find solutions of the form $y = x^r$, $x > 0$.
Plugging in leads to the equation

$$F(r) := r(r-1) + \alpha r + \beta = 0.$$

case 1 : IF $F(r)$ has two distinct real roots then the general solution is $y = C_1 x^{r_1} + C_2 x^{r_2}$, $x > 0$.

case 2: If $F(r)$ has one repeated root then the general solution is $y = (C_1 + C_2 \ln x) x^{r_1}$, $x > 0$.

case 3: IF $F(r)$ has two complex roots, say $r_1 = \lambda \pm \mu i$ then the general solution is $y = C_1 x^\lambda \cos(\mu \ln x) + C_2 x^\lambda \sin(\mu \ln x)$, $x > 0$.

10.1 Two-Point Boundary Value Problems

- Physical applications of ODEs often lead to boundary value problems in which the value of the dependent variable or its derivative is specified at two different points.

- example: $y'' + p(x)y' + q(x)y = g(x)$ } *
 $y(\alpha) = y_0, \quad y(\beta) = y_1.$

Unlike in initial value problems, continuity of $p(x)$, $q(x)$, & $g(x)$ does not guarantee the existence or uniqueness of *.

10.2 Fourier Series

- Certain Periodic Functions can be expressed as an infinite trigonometric series called a Fourier series.

- A function f is said to be periodic with period $T > 0$ if the domain of f contains $x+T$ whenever it contains x and if

$$f(x+T) = f(x) \quad \text{for all } x.$$

- The Fourier series expansion of f is given by ↑ defined on $[-L, L]$

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right).$$

Where



$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \quad \text{and}$$

$\uparrow n=0,1,2,\dots$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n=1,2,3,\dots$$