MATH 2930, Fall 2018
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Sections: 212, 217

Name:SOLUTIONS

Chapter 1.1 Review

Objectives: (1) To motivate the study of differential equations through an example. (2) To practice sketching direction fields. (3) To make qualitative statements about differential equations without solving them, i.e. finding equilibrium solutions, and long time behavior.

- Equations containing derivatives are called differential equations.
- In order to study physical problems such as motion of fluids, the flow of current in electric circuits, increase or decrease of populations it is necessary to know something about differential equations (there are many, you only needed to think of three).

Motivating Example: Suppose that an object is falling in the atmosphere near sea level. Formulate a differential equation that describes its motion (*Hint: Recall Newton's second law F=ma. What are the forces acting on the object as it falls?*).

Let t denote time and $v(t)$ represent the velocity of the falling object as a function of time. Newtons second law tells us that the force acting on the object is equal to its mass times its acceleration. Mathematically, this means

$$F = ma.$$

Since the a is related to v by $a = dv/dt$, we actually have

$$F = m\frac{dv}{dt}.$$

Next, we consider the forces acting on the object. Gravity exerts a downward force given by mg where g is the acceleration due to gravity. There is also a force due to air resistance called the drag force which is proportional to the objects velocity. This is given by γv where γ is a constant called the drag coefficient. Combining these two forces we see that

$$F = mg - \gamma v$$

Hence, we obtain the following differential equation

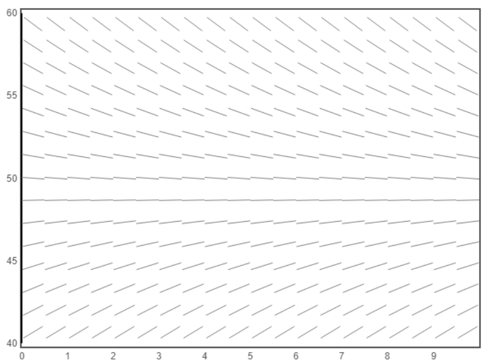
$$m\frac{dv}{dt} = mg - \gamma v.$$

Direction Fields: Consider the differential equation that you found in the previous example. For simplicity, lets assume that that $m = 10$kg (mass of object), $\gamma = 2$kg/s (drag force). Your differential equation should then become

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}. \tag{1}$$

Suppose that the velocity v has a certain given value. Then, by evaluating the right-hand side of equation (1), we can find the corresponding value $\frac{dv}{dt}$. For example, if $v = 40$, then $\frac{dv}{dt} = 1.8$. This means that the slope of a solution $v(t)$ has the value 1.8 at any time where $v = 40$. A sketch of the slopes for various values of v and t in the $tv-$plane is called a direction field.

Sketching Direction Field: Sketch a direction field for equation (1).



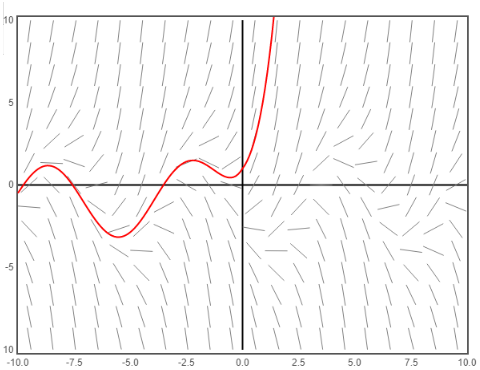
A solution of a differential equation that does not change with time is called an equilibrium solution.

Qualitative Analysis: Using your direction field as a guide, find an equilibrium solution to equation (1). Based off of your sketch, how does a solution to (1) behave as $t \to \infty$? What do you think is a physical interpretation of an equilibrium solution? (*Hint: In the context of this problem, the equilibrium solution is sometimes called the terminal velocity*).

By setting $\frac{dv}{dt} = 0$, one can check that the constant function $v(t) = 49$ is an equilibrium solution to (1). This solution corresponds to the perfect balance between the gravity and the drag force. Based off of your sketch you may notice that all other solutions appear to converge to the equilibrium solution.

Additional Excersies:

1. Sketch a direction field for $y' = 3\sin t + 1 + y$.



Note that the red curve is a guess of a possible solution $y(t)$ based off of the direction field.

2. A population of insects in a region will grow at a rate that is proportional to their current population. On any given day there is a net migration into the area of 15 insects and 16 are eaten by the local bird population and 7 die of natural causes. Write down a differential equation that models the population at a given time. Assume that the proportionality constant for the birth rate is .2 and that there are initially 100 insects in the area. Sketch a direction field to try to determine whether the population will survive or not?

Let $P(t)$ denote the population as a function of time. Then,

$$\frac{dP}{dt} = \text{birth rate} + \text{migration in} - \text{mortality rate} - \text{migration out}.$$

The first part of the question tells use that

$$\text{birth rate} = kP(t)$$

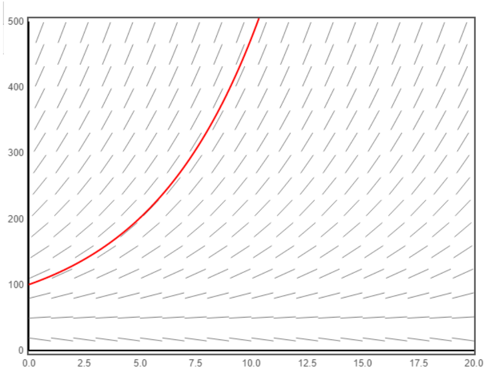
for some positive constant k. Therefore, combing these two we have

$$\frac{dP}{dt} = (kP(t) + 15) - (16 + 7) = kP(t) - 8.$$

Assuming that the proportionality constant is .2 and that there are 100 insects initially we obtain the following differential equation:

$$\frac{dP}{dt} = .2P(t) - 8.$$

with $P(0) = 100$. A direction field for this differential equation near the initial value is:



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This implies that the population of the insects will survive. The curve in red represents a feasible solution for the equation with the given initial condition.