## Chapter 1.1, 1.3, 2.1 Review

Objectives: (1) Identify dependent vs. independent variables in differential equations. (2) Draw direction fields (3) Recognize and solve simple separable ODEs (4) Recognize and solve 1st order linear ODEs with integrating factors

Part 1. For the differential equations below, do the following: state the dependent and independent variables, state the order of the equation, and determine whether it is linear or nonlinear.

1. $\left(1+y^{2}\right) \frac{d^{2} y}{d t^{2}}+t \frac{d y}{d t}+y=e^{t}$
$y$ is the dependent variable, $t$ is the independent variable. The equation is second order and it is nonlinear becaues of the presence of the $y^{2}$ term.
2. $y^{\prime \prime \prime \prime}+4 y^{\prime \prime \prime}+3 y=x$
$y$ is the dependent variable, $x$ is the independent variable. The equation is fourth order. It is linear.
3. $\frac{d^{2} g}{d x^{2}}+\sin (x+g)=\sin x$
$g$ is the dependent variable, $x$ is the independent variable. The equation is second order and it is nonlinear because of $g$ is found inside the $\sin (x+g)$.
4. $f^{\prime}=f(f-3)$
$f$ is the dependent variable. The independent variable is not present in this notation. The equation is first order and it is nonlinear because there is an $f^{2}$ term present.

Part 2. Sketch a direction field for the following differential equations.

1. $y^{\prime}=y(y+3)$


Part 3. State the general solving of the following general first order linear differential equation (Hint: integrating factor).

$$
\frac{d y}{d t}+p(t) y=g(t)
$$

Let

$$
\mu(t)=e^{\int p(t) d t}
$$

then the general solution is given by

$$
y(t)=\frac{1}{\mu(t)}\left(\int \mu(t) g(t) d t+C\right) .
$$

Part 4. State whether each of the differential equations below are separable or not. Then, solve.

1. $y^{\prime}+3 y=t+e^{-2 t}$

This equation is not separable. We will use the method of integrating factors to solve. The integrating factor is $e^{\int 3 d t}=e^{3 t}$. So, $\left(e^{3 t} y(t)\right)^{\prime}=t e^{3 t}+e^{t}$. Integrating both sides:

$$
e^{3 t} y(t)=\frac{t e^{3 t}}{3}-\frac{e^{3 t}}{9}+e^{t}+C
$$

So, the general solutions is

$$
y(t)=\frac{t}{3}+e^{-2 t}-\frac{1}{9}+C e^{-3 t}
$$

Note that $y$ asymptotic to $t / 3-1 / 9$ as $t \rightarrow \infty$.
2. $y^{\prime}+\frac{2}{t} y=\frac{\cos t}{t^{2}}, y(\pi)=0, t>0$

This equation is not separable. We will use the method of integrating factors to solve. The integrating factor is $e^{\int 2 / t d t}=e^{2 \ln |t|}=t^{2}$. So, $\left(t^{2} y(t)\right)^{\prime}=t^{2} \times \frac{\cos t}{t^{2}}=\cos t$. Integrating both sides yields,

$$
t^{2} y(t)=\sin t+C
$$

So, the general solution is

$$
y(\pi)=\frac{\sin t+C}{t^{2}}
$$

Using the initial condition we find that $C=0$. So the final answer is

$$
y(t)=\frac{\sin t}{t^{2}}
$$

3. $y^{\prime}=\frac{x^{2}}{y}$

This equation is separable. The equation can be written

$$
\frac{d y}{d x}=\frac{x^{2}}{y}
$$

Separating the $y$ and $x$ terms to opposite sides gives:

$$
y d y=x^{2} d x
$$

Integrating both sides with respect to the corresponding variables gives:

$$
\int y d y=\int x^{2} d x \Longrightarrow \frac{y^{2}}{2}=\frac{x^{3}}{3}+C
$$

Hence we get the following implicitly defined solution for $y \neq 0$ :

$$
2 y^{2}-2 x^{3}=c
$$

4. $x y^{\prime}=\left(1-y^{2}\right)^{1 / 2}$

First note that $y= \pm 1$ are the constant solutions. Otherwise, we can separate and integrate to obtain:

$$
\begin{aligned}
& \int \frac{d y}{\left(1-y^{2}\right)^{1 / 2}}=\int \frac{d x}{x} \Longrightarrow \\
& \int \frac{d y}{\left(1-y^{2}\right)^{1 / 2}}=\ln |x|+C
\end{aligned}
$$

If $|y|<1$ then we can make the substitution $y=\sin \theta$ for $-\pi / 2 \leq \theta \leq \pi / 2$. Then, the integral on the left becomes:

$$
\int d \theta=\theta=\sin ^{-1}(y) .
$$

So, the general solution is

$$
y(x)=\sin (\ln |x|+C),|y|<1, x \neq 0 .
$$

Part 5. Find the value $y_{0}$ for which the solution of the initial value problem below remains finite as $t \rightarrow \infty$.

- $y^{\prime}-y=1+3 \sin t, y(0)=y_{0}$.

We will first solve the equation using the method of integrating factors. The integrating factor is $e^{\int}-1 d t=$ $e^{-t}$. So

$$
\left(e^{-t} y(t)\right)^{\prime}=e^{-t}+3 e^{-t} \sin t
$$

Integrating both sides we obtain:

$$
e^{-t} y(t)=-e^{-t}+3 \int e^{-t} \sin t d t
$$

The remaining integral can be evaluated by integration by parts twice in a row:

$$
\int e^{-t} \sin t d t=-e^{-t} \cos t-\int e^{-t} \cos t d t=-e^{-t} \cos t-\left(e^{-t} \sin t+\int e^{-t} \sin t d t .\right)
$$

So, solving for the desired integral we get:

$$
\int e^{-t} \sin t d t=\frac{-e^{-t}(\cos t+\sin t)}{2}
$$

Hence, the general solution is:

$$
y(t)=-1-\frac{3(\cos t+\sin t)}{2}+C e^{t}
$$

Using the initial condition $y(0)=y_{0}$ we get:

$$
y(t)=-1-\frac{3(\cos t+\sin t)}{2}+\left(y_{0}+\frac{5}{2}\right) e^{t} .
$$

As $t \rightarrow \infty$ the last term goes to infinity. So, in order to obtain finite solutions as $t \rightarrow \infty$ we need $y_{0}=-\frac{5}{2}$.

