MATH 2930, Fall 2018
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Sections: 212, 217 **Name:**SOLUTIONS

Chapter 2.2 and 2.3 Review

Objectives: (1) Identity and solve separable differential equations (2) Practice modeling with first-order differential equations

Part 1- Separable Differential Equations.

1. Solve the differential equation
$$\frac{dy}{dx} = \frac{-x}{y}.$$
We write: $ydy = -xdx$. Integrating both sides, we obtain: $y^2/2 = -x^2/2 + C$. So, we obtain the following general solutions:
$$y = \pm\sqrt{-x^2 + c}.$$
The initial condition would be used to determine whether the positive or negative solution is to be used and also to determine the interval in which the solution is defined.

2. Solve the initial value problem
$$y' = \frac{2 - e^x}{3 + 2y}, y(0) = 0$$
and determine where the solution attains its maximum value.
We write: $3y + y^2 = 2x - e^x + C$. Using the initial condition we see that $C = 1$. Next, we can use the quadratic formula to obtain y explicitly
$$y = \frac{-3 \pm \sqrt{13 + 8x - 4e^x}}{2}.$$
In order to see if we should choose the "+" or the "-" we can use the initial condition again, note that
$$y(0) = \frac{-3 \pm \sqrt{9}}{2} = \frac{-3 \pm 3}{2} = 0.$$
This is only true if we choose +. So, finally we get that
$$y(x) = \frac{-3 + \sqrt{13 + 8x - 4e^x}}{2}.$$
To see where the function attains its maximum we check which values of x make $dy/dx = 0$. By looking at the differential equation, we see that happens for $x = \ln 2$. In order to verify that this is indeed a maximum we can differentiate the differential equation again using the product rule to get:
$$\frac{d^2y}{dx^2} = \frac{-e^x}{3 + 2y} + (2 - e^x)\frac{d}{dx}\left(\frac{1}{3 + 2y(x)}\right).$$
We leave the derivative in the second term unsimplified. Evaluating at $x = \ln 2$, we get:
$$\left.\frac{d^2y}{dx^2}\right|_{x=\ln 2} = \frac{-2}{3 + 2y}.$$
Using the expression for y we see that $y(x) \geq -3/2$. In fact, since $\frac{dy}{dx}$ is undefined at $y = -3/2$, actually have that $y(x) > -3/2$ for all x where y is defined. Hence the term $\frac{1}{3+2y} > 0$ for all x where the solution is defined. This combined with the previous computation shows that $y''(\ln 2) < 0$. So, $x = \ln 2$ is the value where y attains its max.

3. Solve the initial value problem
$$y' = \frac{1 + 3x^2}{3y^2 - 6y}, y(0) = 1$$
and determine the interval in which the solution is valid. Hint. To find the interval of definition, look for points where the integral curve has a vertical tangent.

We have $3y^2 - 6ydy = 1 + 3x^2dx$. Integrating both sides, we get

$$y^3 - 3y^2 = x + x^3 + C.$$

Using the initial condition, we see that $C = -2$.So, we have that $y^3 - 3y^2 = x + x^3 - 2$. In order to find the interval of convergence, we note that $y' \to \infty$ as $y \to 0$ and as $y \to 2$. The x values corresponding to these y values are ± 1. So, the interval in which the solution is valid is $(-1, 1)$.

Part 2- Modeling with First Order Differential Equations.

1. Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200L of a dye solution with a concentration of 1 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2L/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.

   Let $y(t)$ be the amount of dye in the tank at time t. Then,

   $$\frac{dy}{dt} = \text{dye coming in per min} - \text{dye going out per min}.$$

   The amount of dye coming in is 0 and the amount of dye going out per minute is

   $$\frac{2y(t)}{200} = \frac{y(t)}{100}.$$

   So, the differential equation is

   $$\frac{dy}{dt} = -\frac{y(t)}{100}, y(0) = 200.$$

   This is a separable differential equation with solution

   $$y(t) = 200e^{-t/100}.$$

   We need the time T such that $y(T) = 2$. This amounts to solving $2 = 200e^{-T/100}$ and we obtain that $T = -100 \ln 1/100$.

2. A young person with no initial capital invests k dollars per year at an annual rate of return r. Assume that investments are made continuously and that the return is compounded continuously.

   (a) Determine the sum $S(t)$ accumulated at any time t.

   (b) If $r = 7.5\%$, determine k so that \$1 million will be available for retirement in 40 years.

   (a) Let $S(t)$ be the amount of money in the person's investment pool at time t. Then,

   $$\frac{dS}{dt} = (\text{amount returned yearly from investments}) + (\text{amount invested yearly}) - (\text{amount losing yearly}).$$

   The amount of money gaining yearly is $rS(t)$ and the amount of money invested yearly is k. So, the differential equation becomes:

   $$\frac{dS}{dT} = rS(t) + k, S(0) = 0.$$

   Note that this is a first order linear differential equation that can be rewritten as follows:

   $$\frac{dS}{dt} - rS(t) = k.$$

   Hence, we can use the method of integrating factors to solve. The integrating factor is $e^{\int -rdt} = e^{-rt}$. So, $(e^{-rt}S(t))' = ke^{-rt}$. Integrating both sides and dividing by e^{-rt} gives:

   $$S(t) = -\frac{k}{r} + Ce^{rt}.$$

   Using the initial condition $S(0) = 0$ we get that $C = k/r$. So, the final solution is

   $$S(t) = \frac{k}{r}(e^{rt} - 1).$$

(b) Plug in $r = .075, t = 40$, and $S = 1,000,000$. Then, solve for k. We get,

$$k = \frac{(.075)(1,000,000)}{(e^{(.075)(40)} - 1)} = \frac{75,000}{e^3 - 1} \approx 3930.$$

3. A tank contains 100 gal of water and 50 oz of salt. Water containing a concentration of $\frac{1}{4}\left(1 + \frac{1}{2}\sin t\right)$ oz/gal flows into the tank at a rate of 2 gal/min, and the mixture in the tank flows out at the same rate. Find the amount of salt in the tank at any time.

   Let $y(t)$ be the amount of salt in the tank. Salt enters the tank of water at a rate of $2(1/4)(1+(1/2)\sin t)$ oz/min. It leaves the tank at a rate of $2y(t)/100$ oz/min. Hence the differential equation governing the amount of salt at any time is

$$\frac{dy}{dt} = \frac{1}{2} + \frac{1}{4}\sin t - \frac{y(t)}{50}, y(0) = 50oz.$$

   This is a first order linear differential equation and the integrating factor is $\mu(t) = e^{t/50}$. Write the equation as $(e^{t/50}y(t))' = e^{t/50}(1/2+(1/4)\sin t)$. The specific solution is $y(t) = 25+(12.5\sin t - 625\cos t + 63150e^{-t/50})/2501$ oz.