MATH 2930, Fall 2018
TA: Aleksandra Niepla
Sections: 212, 217

Name:SOLUTIONS

Chapter 2.4, 2.5, 2.6 Review

Objectives: (1) Review theorems regarding existence and uniqueness for first order linear equations (2) Sketch phase lines and classify equilibrium solutions of autonomous differential equations (3) Solve exact differential equations

Part 1- Existence and Uniqueness Theorems

1. Consider a first order linear equation of the form

$$y' + p(t)y = g(t), \qquad y(t_0) = y_0.$$

Summarize the hypotheses needed to guarantee the existence and uniqueness of a solution to the above equation.

If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique solution $y = \phi(t)$ that satisfies the initial value problem on the interval I.

2. Do the same the following general first order non-linear equation:

$$y' = f(t, y), \qquad y(t_0) = y_0.$$

If the functions f and $\partial f/\partial y$ are continuous in some rectangle R: $\alpha < t < \beta, \gamma < y < \delta$ containing the point (t_0, y_0), then there exists a unique solution $y = \phi(t)$ of the initial value problem above, defined in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$.

3. Without solving the problem, determine an interval in which a unique solution of the given initial value problem is certain to exist.

$$(4 - t^2)y' + 2ty = 3t^2, \qquad y(-3) = 1.$$

The equation can be rewritten as:

$$y' + \frac{2t}{4 - t^2}y = \frac{3t^2}{4 - t^2}.$$

The functions corresponding to $p(t)$ and $g(t)$ in the statement above are continuous on the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$ since we have to avoid the points ± 2 which is where the denominator is zero. Since the interval must contain the initial value $t_0 = -3$, we conclude that there exists a unique solution to the initial value problem that is defined on the interval $(-\infty, -2)$.

Part 2- Autonomous Differential Equations and Phase Line Sketches

1. An important class of first order equations consists of those in which the independent variable does not appear explicitly, i.e.

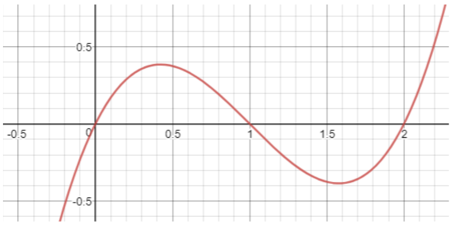
$$\frac{dy}{dt} = f(y).$$

Such equations are called autonomous differential equations.

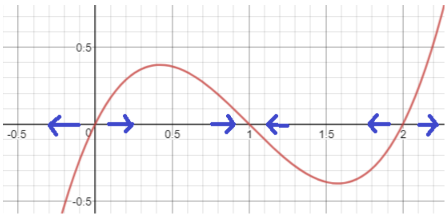
2. In the problem below, sketch the graph of $f(y)$ versus y, determine the equilibrium solutions (critical points), and classify each one as asymptotically stable or unstable.

$$\frac{dy}{dt} = y(y - 1)(y - 2), \qquad y_0 \geq 0.$$

The sketch of $f(y)$ versus y is:

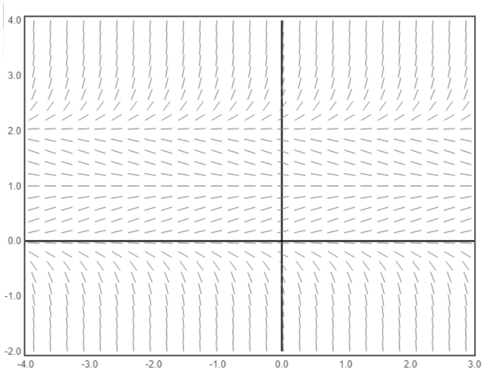


Along the $y-axis$, we draw a right arrow if $f(y)$ is positive and a left facing arrow if $f(y)$ is negative. Note that the phase line sketch is just the rotated version of the y axis indicating the equilibrium points and the arrows as on p. 66 of the text book.



Equilibrium solutions with arrows pointing towards them are stable and ones with arrows pointing away from them are unstable. Hence, $y=0$ and $y=1$ are asymptotically unstable equilibrium solutions and $y=2$ is asymptotically stable.

Note how these graphs related to a direction field sketch of the equation:



Solutions diverge away from the equilibrium solutions 0 and 2 and converge towards 1 if the initial condition is in $(0,2)$.

Part 3- Exact Differential Equations

1. Let M, N, M_y, and N_x be continuous in the rectangular region R given by $\alpha < x < \beta, \gamma < y < \delta$. Then, the equation

$$M(x,y) + N(x,y)y' = 0$$

Is an **exact differential equation** in R if and only if $\underline{M_y(x,y) = N_x(x,y)}$

at each point of R. That is, there exists a function ψ satisfying

$$\psi_x(x,y) = M(x,y) \qquad\qquad \psi_y(x,y) = N(x,y).$$

The solution is given implicitly as

$$\psi(x,y) = c.$$

2. Solve the given initial value problem

$$(2x - y) + (2y - x)y' = 0, \qquad y(1) = 3.$$

In this case, we have $M(x, y) = 2x - y$ and $N(x, y) = 2y - x$. So, $M_y = -1 = N_x$. Therefore, the equation is exact. Moreover, we know that the general solution is $\psi = c$ where ψ is a function that satisfies

$$\psi_x(x, y) = M(x, y) \qquad \text{and} \qquad \psi_y(x, y) = N(x, y).$$

The first tells us that

$$\psi(x, y) = \int 2x - ydx = x^2 - xy + h(y).$$

Similarly the second tells us that

$$\psi(x, y) = \int 2y - xdy = y^2 - xy + g(x).$$

Hence, we must take $h(y) = y^2$ and $g(x) = x^2$. Finally, we get that the general solution has the form:

$$x^2 + y^2 - xy = c.$$

Using the initial condition we get:

$$x^2 + y^2 - xy = 7.$$

We can solve for y explicitly using the quadratic formula to obtain:

$$y = \frac{x + \sqrt{28 - 3x^2}}{2}.$$

Note that the initial condition is used again here to help determine which sign to take in front of the radical.