MATH 2930, Fall 2018
TA: Aleksandra Niepla
Sections: 212, 217

Name:____________________________

Chapter 2.7, 3.1, 3.3 Review

Objectives: (1) Approximate solutions to first order differential equations using Euler's method (2) Practice solving second order linear homogeneous differential equations with constant coefficients

Part 1- Numerical Approximations: Euler's Method

1. Consider the initial value problem

$$y' = 0.5 - t + 2y, \qquad y(0) = 1$$

(i) Find approximate values of the solution of the given initial value problem at $t = 0.1, 0.2, 0.3,$ and 0.4 using Euler's method with $h = 0.1$.
(ii) Find the solution $y(t)$ of the given problem and evaluate $y(t)$ at $t = 0.1, 0.2, 0.3,$ and 0.4. Compare your results with those of part (i).

(i) We have the following approximations:

n	t_n	y_n	$f(t_n, y_n)$	Line approximation at point (t_n, y_n) $y = y_n + f(t_n, y_n)(t - t_n)$
0	0	1	2.5	$y = 2.5\,t + 1$
1	0.1	1.25	2.9	$y = 1.25 + 2.9\,(t - 0.1)$
2	0.2	1.54	3.38	$y = 1.54 + 3.38\,(t - 0.2)$
3	0.3	1.878	3.956	$y = 1.878 + 3.956\,(t - 0.3)$
4	0.4	2.2736		

(ii) Rewriting the equation we have:

$$y' - 2y = 0.5 - t.$$

Let $\mu(t) = e^{\int -2dt} = e^{-2t}$. Then

$$(e^{-2t}y(t))' = 0.5e^{-2t} - te^{-2t}.$$

Integrating both sides and dividing by e^{-2t} we get:

$$y(t) = \frac{t}{2} + Ce^{2t}.$$

The initial condition implies that $C = 1$. So, the solutions is:

$$y(t) = t/2 + e^{2t}.$$

Moreover, using a calculator we have:

$$y(0.1) = 1.27, y(0.2) = 1.59, y(0.3) = 1.97, y(0.4) = 2.43.$$

Part 2- Second Order Linear Differential Equations

1. Find the general solution of the given differential equation

$$y'' + 2y' - 3y = 0.$$

The characteristic equation is

$$r^2 + 2r - 3 = 0.$$

We factor to find its roots:

$$r^2 + 2r - 3 = 0 \implies (r+3)(r-1) = 0 \implies r_1 = -3, r_2 = 1.$$

So, the general solutions is

$$y(t) = c_1 e^{-3t} + c_2 e^t.$$

2. Find the solution of the given initial value problem. Describe its behavior as t increases:

$$y'' + 8y' - 9y = 0 \qquad y(1) = 1, y'(1) = 0.$$

The characteristic equation is

$$r^2 + 8r - 9 - 0.$$

We factor to find its roots

$$r^2 + 8r - 9 = 0 \implies (r+9)(r-1) = 0 \implies r_1 = -9, r_2 = 1.$$

Therefore, the general solutions is

$$y(t) = c_1 e^{-9t} + c_2 e^t.$$

Using the two initial conditions provided we obtain the following system of equations:

$$c_1 e^{-9} + c_2 e = 1$$

and

$$-9c_1 e^{-9} + c_2 e = 0$$

Multiplying the first by 9 and adding to the second gives:

$$10ec_2 = 9 \implies c_2 = \frac{9}{10} e^{-1}.$$

Next, we can plug c_2 into the first equation and solve for c_1 :

$$c_1 e^{-9} + \frac{9}{10} = 1 \implies c_1 = \frac{1}{10} e^9.$$

So, the final solutions is:

$$y(t) = \frac{1}{10} e^{-9(t-1)} + \frac{9}{10} e^{t-1}; y \to \infty \quad \text{as} \quad t \to \infty.$$

3. Find the general solution of the given differential equation

$$y'' + 6y' + 13y = 0.$$

The characteristic equation is

$$r^2 + 6r + 13 = 0.$$

The roots are:

$$r = \frac{-6 \pm \sqrt{36 - 4(13)}}{2} = \frac{-6 \pm \sqrt{36 - 52}}{2} = \frac{-6 \pm \sqrt{-16}}{2} = -3 - 2i, -3 + 2i.$$

So, the general solutions is

$$y(t) = c_1 e^{-3t} \cos(2t) + c_2 \sin(2t) e^{-3t}.$$

4. Determine the values of α, if any, for which all solutions tend to zero as $t \to \infty$; also determine the values of α, if any, for which all nonzero solutions become unbounded as $t \to \infty$ for

$$y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0.$$

The characteristic equation is:

$$r^2 - (2\alpha - 1)r + \alpha(\alpha - 1) = 0.$$

Using the quadratic formula, the roots of this equations are:

$$r = \frac{(2\alpha - 1) \pm \sqrt{(2\alpha - 1)^2 - 4(\alpha(\alpha - 1))}}{2} = \frac{(2\alpha - 1) \pm \sqrt{4\alpha^2 - 4\alpha + 1 - 4\alpha^2 + 4\alpha}}{2} = \frac{(2\alpha - 1 \pm 1)}{2} = \alpha, \alpha - 1$$

So, the general solution to the equation is:

$$y(t) = c_1 e^{\alpha t} + c_2 e^{(\alpha - 1)t}.$$

Assuming that $\alpha \in \mathbb{R}$, all solutions will tend to zero as long as $\alpha < 0$. On the other hand, all solutions will become unbounded as long as $\alpha - 1 > 0$, that is $\alpha > 1$.

5. Consider the equation $ay'' + b'y + cy = 0$, where a, b, and c are constants with $a > 0$. Find the conditions on a, b, and c such that the roots of the characteristic equation are:
(i) real, different, and negative.
(ii) real with opposite signs.
(iii) real, different, and positive.
In each case, determine the behavior of the solutions as $t \to \infty$.

The characteristic equation is

$$ar^2 + br + c = 0.$$

Using the quadratic formula the roots are:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(i) In order for the roots to be real and different we need $b^2 - 4ac > 0$. This gives us the relation: $c < b^2/2a$. Since $a > 0$, it is the numerator that determines the sign of the roots. Since we want them to be negative, we need:

$-b \pm \sqrt{b^2 - 4ac} < 0 \implies b^2 - 4ac < b^2$. Since a is positive this only happens when $c > 0$. Finally, we need $b > 0$. So the conditions are: $b > 0$, $0 < c < b^2/4a$. All solutions of this type will go to zero as $t \to \infty$.

(ii) To be real with opposite signs, we still need $b^2 - 4ac > 0$. If $b > 0$ then you know $-b - \sqrt{b^2 - 4ac} < 0$ and you would need $-b + \sqrt{b^2 - 4ac} > 0 \implies c < 0$. If $b < 0$, then, a similar analysis shows $c < 0$. So, the only condition we need is $c < 0$. All solutions of this type become unbounded as $t \to \infty$.

(iii) To be real and different we need $c < b^2/2a$. For both of them to have positive signs we see right away that we need $b < 0$. This are the only two restrictions in this case. Solutions of this type become unbounded as $t \to \infty$.

6. Solve the given equation for $t > 0$:

$$t^2 y'' + 4ty' + 2y = 0.$$

Hint: Let $x = \ln t$ and calculate dy/dt and d^2y/dt^2 in terms of dy/dx and d^2y/dx^2.

Let $x = \ln(t)$. Then,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx}\frac{1}{t}.$$

and

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{1}{t} - \frac{dy}{dx}\frac{1}{t^2} = \frac{d^2y}{dx^2}\frac{1}{t^2} - \frac{dy}{dx}\frac{1}{t^2} = \frac{1}{t^2}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right).$$

Plugging this into the differential equation we obtain:

$$\frac{d^2}{dx^2} - \frac{dy}{dx} + 4\frac{dy}{dx} + 2y = 0.$$

This is now a homogeneous second order differential equation with characteristic equation: $r^2 + 3r + 2 = 0$. The roots are $r = -1, -2$. So the general solution is:

$$y(x) = c_1 e^{-x} + c_2 e^{-2x}.$$

Finally, using that $x = \ln(t)$, we get

$$y(t) = c_1 t^{-1} + c_2 t^{-2}.$$