

Variation of Parameters

1 Find the general solution of the following differential equation. When finding a particular solution first use the method of undetermined coefficients, and then use the method of variation of parameters.

$$1. y'' - 5y' + 6y = 2e^t.$$

Characteristic Equation :

$$r^2 - 5r + 6 = 0 \Rightarrow r = \frac{+5 \pm \sqrt{25 - 4(1)(6)}}{2}$$

$$= \frac{+5 \pm \sqrt{25 - 24}}{2}$$

$$= \frac{+5 \pm 1}{2}$$

$$= \frac{+5 \pm 1}{2}.$$

$$\text{So, } r_1 = 3 \text{ & } r_2 = 2$$

The homogeneous solution is $y_h(t) = C_1 e^{-3t} + C_2 e^{2t}$.

Using the methods of undetermined coefficients to find a particular solution:

$$\text{Let } y_p(t) = Ae^t \rightarrow$$

Then, plugging in gives:

$$Ae^t - 5Ae^t + 6Ae^t = 2e^t \Rightarrow$$

$$2Ae^t = 2e^t \Rightarrow A = 1.$$

So, the general solution is:

$$y(t) = c_1 e^{3t} + c_2 e^{2t} + e^t$$

$\uparrow \quad \uparrow$
 $y_1 \quad y_2$

Method of variation of parameter :

We replace the constants of the general solution to the homogeneous problems by functions:

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t). \quad (\star)$$

We try to determine u_1 & u_2 so that (\star) is a solution to the nonhomogeneous problem.

$$\begin{aligned} \text{Differentiating } (\star) : \quad y'(t) &= u_1'(t)y_1(t) + u_1(t)y_1'(t) \\ &\quad + u_2'(t)y_2(t) + u_2(t)y_2'(t). \end{aligned}$$

For simplicity assume that $u_1'(t)y_2(t) + u_2'(t)y_1(t) = 0$



Then, we have :

$$y' = u_1(t)y_1'(t) + u_2(t)y_2'(t).$$

Differentiating again:

$$y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''.$$

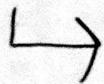
Substituting y, y', y'' back into the equation:

$$\begin{aligned} & u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' - 5(u_1(t)y_1'(t) + \\ & \qquad \qquad \qquad u_2(t)y_2'(t)) \\ & + 6(u_1(t)y_1(t) + u_2(t)y_2(t)). \end{aligned}$$

Rearranging:

$$\begin{aligned} & u_1(t)[y_1'' - 5y_1' + 6y_1] + \downarrow \text{is zero} \\ & u_2(t)[y_2'' - 5y_2' + 6y_2] + \swarrow \text{is zero} \\ & u_1'y_1' + u_2'y_2' = 2e^t. \end{aligned}$$

$$\Rightarrow u_1'y_1' + u_2'y_2' = 2e^t$$



This together with the assumption we made previously gives us the system of equations:

$$\begin{cases} u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0 \\ u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = 2e^t \end{cases}$$

Solving the system we get (here we use cramer's rule)

$$u_1' = \frac{\begin{bmatrix} 0 & y_2(t) \\ 2e^t & y_2'(t) \end{bmatrix}}{\det \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}}, \quad u_2' = \frac{\begin{bmatrix} y_2(t) & 0 \\ y_2'(t) & 2e^t \end{bmatrix}}{\det \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}}$$

$$= -\frac{2y_2(t)e^t}{y_1y_2' - y_2y_1'} = \frac{2y_1(t)e^t}{y_2y_1' - y_1y_2'}$$

$$\Rightarrow u_1 = - \int \frac{2e^t \cdot e^{2t}}{(-e^{5t})} dt, \quad u_2 = \int \frac{2e^t e^{3t}}{(-e^{5t})} dt$$

$$u_1 = +2 \int \frac{e^{3t}}{e^{5t}} dt = +2 \int e^{-2t} dt = -e^{-2t}$$

$$u_2 = 2 \int \frac{e^{4t}}{(-e^{5t})} dt = -2 \int e^{-t} dt = +2e^{-t}.$$

$$\begin{aligned} \text{So, } Y(t) &= -e^{3t} \cdot e^{-2t} + 2e^{-t} \cdot e^{2t} \\ &= -e^t + 2e^t = e^t. \quad \text{same as before} \checkmark \end{aligned}$$

2. The characteristic equation is:

$$4r^2 - 4r + 1 = 0.$$

$$\text{So, } r = \frac{4 \pm \sqrt{16 - (4)(4)(1)}}{8} = \frac{1}{2}. \text{ So } y_1(t) = e^{\frac{1}{2}t}$$

This is a repeated root. So, $y_2(t) = te^{\frac{1}{2}t}$ is another solution. From before we know that $Y(t) = u_1(t)y_1 + u_2(t)y_2$ is a particular solution where

$$u_1 = - \int \frac{y_2(t) 4e^{\frac{1}{2}t}}{\det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}} dt \quad u_2 = \int \frac{y_1(t) 4e^{\frac{1}{2}t}}{\det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}} dt \quad \hookrightarrow$$

$$\det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det \begin{bmatrix} e^{1/2t} & te^{1/2t} \\ \frac{e^{1/2t}}{2} & e^{1/2t} + \frac{te^{1/2t}}{2} \end{bmatrix}$$

$$= e^t + \frac{te^t}{2} - \frac{te^{1/2t}}{2}$$

$$= e^t.$$

$$\text{So, } u_1 = -4 \int \frac{te^t}{e^t} dt = -\frac{4t^2}{2} = -2t^2.$$

$$u_2 = \int \frac{4e^t}{e^t} dt = 4t.$$

$$\text{So, } y(t) = -2t^2 e^{1/2t} + 4t^2 e^{1/2t}$$

$$= 2t^2 e^{1/2t}.$$

So, the general solution is

$$y(t) = C_1 e^{1/2t} + C_2 t e^{1/2t} + 2t^2 e^{1/2t}$$

3. Find the general solution to the equation
 $y'' + y = \tan t.$

The homogeneous problem has characteristic equation

$$r^2 + 1 = 0 \Rightarrow r^2 = -1 \Rightarrow r = \pm i.$$

So, two linearly independent solutions to
the homogeneous problem are $y_1(t) = \cos t$
 $y_2(t) = \sin t.$

Then, a particular solution is

$$y_p(t) = u_1(t) \cos t + u_2(t) \sin t \text{ where}$$

$$u_1 = - \frac{\int \sin t \cdot \tan t \, dt}{\det \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}} \quad u_2 = \frac{\int \cos t \tan t \, dt}{\det \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}}$$

$$= - \int \frac{\sin t \cdot \tan t}{1} \, dt \quad u_2 = \int \frac{\cos t \sin t}{\cos t} \, dt$$

$$= - \int \frac{\sin t}{\cos t} \, dt \quad = \int \sin t \, dt = -\cos t.$$

Next, we have

$$\begin{aligned} M_1 &= \int \sin t \tan t \, dt = \int \sin t \cdot \frac{\sin t}{\cos t} \, dt \\ &= \int \sin^2 t \cdot \sec t \, dt \\ &= \int (1 - \cos^2 t) \sec t \, dt \\ &= \int \sec t - \cos t \, dt \\ &= \ln |\tan x + \sec x| - \sin x. \end{aligned}$$

So, the general solution is

$$\begin{aligned} y(t) &= C_1 \sin t + C_2 \cos t - \cos t \sin t - \ln |\tan t + \sec t| \cdot \cos t \\ &\quad + \sin t \cos t \end{aligned}$$

$$= C_1 \sin t + C_2 \cos t - \ln |\tan t + \sec t| \cos t.$$

Part 2: Mechanical and Electrical Vibrations

1. List some physical problems whose solutions are all described by solutions of the initial value problem:

$$ay'' + by' + cy = g(t), y(0) = 0, y'(0) = 0$$

a mass on a vibrating spring, the torsional oscillations of a shaft wheel, flow of electric current in a circuit.

2. Describe how the motion of a mass on a spring can be modeled by an equation of the form above.

see solutions 5(A) of prelim #1

3. A mass weighing 4 lb stretches a spring 2 in. Suppose that the mass is given an additional 6 in displacement in the positive direction and then released. The mass is in a medium that exerts a viscous resistance of 6 lb when the mass has a velocity of 3 ft/s. Formulate the initial value problem that governs the motion of the mass.

see p. 149 - 150 of course
text book.

4. Assume that the system described by the differential equation $mu'' + \gamma u' + ku = 0$ is either critically damped or overdamped. Show that the mass can pass through the equilibrium position at most once, regardless of the initial conditions. (Hint: Determine all possible values of t for which $u = 0$).

Critically damped or overdamped means that $\gamma \geq 2\sqrt{km}$. Its equilibrium position is $u(t) = 0$.

Investigate the possible solutions on p. 152,
to finish solving the problem.