MATH 2930, Fall 2018
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Sections: 212, 217

Name:SOLUTIONS

Chapter 4.1-4.3 Review

Objectives: (1) Review the general theory for n^{th} order linear differential equations (2) Solve homogeneous linear equations with constant coefficients (3) Use the method of undetermined coefficients to solve linear equations with constant coefficients

Part 1: General Theory of n^{th} Order Linear Differential Equations

1. Determine the intervals where solutions of the following differential equation are sure to exist:

$$(x^2-4)y^{(6)}+x^2y'''+9y=0.$$

Rewriting the equation in standard form, we have

$$y^{(6)}+\frac{x^2}{(x^2-4)}y'''+\frac{9}{(x^2-4)}y=0.$$

The second two coefficient functions are not continuous at $x=\pm 2$. So, The intervals on which solutions are sure to exist are: $(-\infty,-2),(-2,2)$ and $(2,\infty)$.

2. Let the linear differential operator L be defined by

$$L[y]=a_0y^{(n)}+a_1y^{(n-1)}+...+a_ny,$$

a. Find $L[t^n]$.
b. Find $L[e^{rt}]$.
c. Determine four solutions of the equation $y^{(4)}-5y''+4y=0$. Do you think the four solutions form a fundamental set of solutions? Why?

a. $L[t^n]=a_0(n\cdot(n-1)\cdot(n-2)\cdot...\cdot 1)+a_1(n\cdot(n-1)\cdot(n-2)\cdot...\cdot 2)\cdot t+a_2nt^{n-1}+...+a_nt^n$.
b. $L[e^{rt}]=a_or^ne^{rt}+a_ar^{n-1}e^{rt}+...+a_ne^{rt}$.
c. Using part b. we see that in our case we have $L[e^{rt}]=(r^4-5r^2+4)e^{rt}$. We see that for each r_i satisfying the equation

$$r^4-5r^2+4=0$$

we have a solution of the form $y_i=e^{r_it}$. To find the roots of this equation, set $u=r^2$. Then we get a quadratic equation:

$$u^2-5u+4=(u-4)(u-1)=0.$$

So, we discovered two roots: $r_1=1$ and $r_2=2$. Using this information we can factor (via synthetic division) r^4-5r^2+4 completely to obtain:

$$r^4-5r^2+4=(r-1)(r^3+r^2-4r-4)=(r-1)(r-2)(r^2+3r+2)=(r-1)(r-2)(r+1)(r+2).$$

So, we get the following solutions: $y_1=e^t,y_2=e^{-t},y_3=e^{2t},y_4=e^{-2t}$. Yes, they form a fundamental set of solutions because they are linearly independent functions. To check this you can compute the Wronskian and see that it is never zero.

Part 2: Homogeneous Differential Equations with Constant Coefficients

In each problem below, find the general solution of the given differential equation.

1. $y^{(4)}-4y'''+4y''=0$

The characteristic equation is $r^4-4r^3+4r^2=0$. Factoring completely we obtain:

$$r^2(r^2-4r+4)=r^2(r-2)^2.$$

Hence $r=0$ and $r=2$ are two roots each with multiplicity 2. So, the general solutions is: $y(t)=c_1+c_2t+c_3e^{2t}+c_4te^{2t}$.

2. $y^{(5)} - 3y^{(4)} + 3y''' - 3y'' + 2y' = 0$

The characteristic equation is $r^5 - 3r^4 + 3r^3 - 3r^2 + 2r = r(r^4 - 3r^3 + 3r^2 - 3r + 2) = 0$. By inspection we see that $r_1 = 1$ is a root. So using synthetic division we can factor further to obtain:

$$r(r^4 - 3r^3 + 3r^2 - 3r + 2) = r(r-1)(r^3 - 2r^2 + r - 2)$$

We know that if r is a rational root of the third degree polynomial above then it is a multiple of 2 (See the discussion on p. 176 if you are not sure why this is true). Now, we can factor completely to obtain:

$$r(r-1)(r^3 - 2r^2 + r - 2) = r(r-1)(r-2)(r^2+1) = r(r-1)(r-2)(r-i)(r+i).$$

So, the general solutions is: $y(t) = c_1 + c_2 e^t + c_3 e^{2t} + c_4 \cos(t) + c_5 \sin(t)$.

3. Show that the general solution of $y^{(4)} - y = 0$ can be written as

$$y = c_1 \cos(t) + c_2 \sin(t) + c_3 \cosh(t) + c_4 \sinh(t).$$

The characteristic equation is $r^4 - 1 = 0$. Right away one sees that $r = \pm 1$ are roots. Hence we can factor: $r^4 - 1 = (r-1)(r+1)(r^2+1) = (r-1)(r+1)(r-i)(r+i)$.Therefore, the general solutions is:

$$y(t) = c_1 \cos(t) + c_2 \sin(t) + c_3 e^t + c_4 e^{-t}.$$

To finish the problem, recall that

$$\cosh t + \sinh t = e^t$$

and

$$\cosh t - \sinh t = e^{-t}.$$

Plug this in to the equation above and relabel the constants.

Part 3: The Method of Undetermined Coefficients Revisited

In each of the problems below, determine the general solution of the given differential equation.

1. $y''' - y'' - y' + y = 2e^{-t} + 3$

First, we find the general solution to the homogeneous problem. The characteristic equation is $r^3 - r^2 - r + 1 = 0$. By inspection we see that $r = 1$ is a solution. Hence we can factor fully to obtain:

$$r^3 - r^2 - r + 1 = (r-1)(r^2-1) = (r-1)^2(r+1).$$

Hence the general solution to the homogeneous problem is

$$y_H(t) = c_1 e^t + c_2 t e^t + c_3 e^{-t}.$$

Finally we can write a particular solution as the sum of particular solutions to the equations; $y''' - y'' - y' + y = 2e^{-t}$ and $y''' - y'' - y' + y = 3$.

Hence consider $Y_P(t) = Ate^{-t} + B$. Note that we multiply the first term by t to distinguish it from the solution e^{-t} already found. So, we have

$$Y_P'(t) = A(e^{-t} - te^{-t}), Y_P''(t) = A(-2e^{-t} + te^{-t}), Y_P'''(t) = A(3e^{-t} - te^{-t}).$$

Plugging in gives:

$$4Ae^{-t} + B = 2e^{-t} + 3.$$

Hence $A = 1/2, B = 3$.

Therefore the general solutions is:

$$y(t) = y_H(t) + y_P(t) = c_1 e^t + c_2 t e^t + c_3 e^{-t} + \frac{1}{2} t e^{-t} + 3.$$