MATH 2930, Fall 2018
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Sections: 212, 217

Name:SOLUTIONS

Chapter 5.4, 10.1-10.2 Review

Objectives: (1) To introduce and solve Euler Equations (2) To discuss and solve two-point boundary value problems (3) to introduce Fourier series and find the Fourier series of given functions

Part 1: Solving Euler Equations

Determine the general solution of the given differential equation that is valid in any interval not including the singular point.

1. $x^2y'' - 3xy' + 4y = 0$
   Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r(r-1)-r+1 = r^2-2r+1$. The root is $r = 1$, with multiplicity two. Hence the general solution, for $x \neq 0$, is $y = (c_1 + c_2 \ln|x|)x$.

2. $(x-2)^2y'' + 5(x-2)y' + 8y = 0$
   Substitution of $y = (x-2)^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r^2+4r+8$. The roots are complex, with $r = -2 \pm 2i$. Hence the general solution, for $x \neq 2$, is $y = c_1(x-2)^{-2}\cos(2\ln|x-2|) + c_2(x-2)^{-2}\sin(2\ln|x-2|)$.

3. $4x^2y'' + 8xy' + 17y = 0$
   Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = 4r^2 + 4r + 17$. The roots are complex, with $r = -1/2 \pm 2i$. Hence the general solution, for $x \neq 0$, is $y = c_1x^{-1/2}\cos(2\ln x) + c_2x^{-1/2}\sin(2\ln x)$.

Part 2: Two-Point Boundary Value Problems

In each problem below, either solve the given boundary value problem or else show that it has no solutions.

1. $y'' + 2y = 0, \;\; y'(0) = 1, y'(\pi) = 0.$
   The characteristic equation is $r^2 + 2 = 0 \implies r = \pm i\sqrt{2}$. So, the general solution is

   $$y(x) = c_1\cos(\sqrt{2}x) + c_2\sin(\sqrt{2}x).$$

   The first boundary condition implies

   $$y'(0) = c_2\sqrt{2} = 1 \implies c_2 = \frac{1}{\sqrt{2}}.$$

   The second boundary condition implies

   $$y'(\pi) = -c_1\sqrt{2}\sin(\sqrt{2}\pi) + \cos(\sqrt{2}\pi) = 0 \implies c_1 = \frac{\cot(\sqrt{2}\pi)}{\sqrt{2}}.$$

   So, the solutions is:

   $$y(x) = \frac{1}{\sqrt{2}}\cot(\sqrt{2}\pi)\cos(\sqrt{2}x) + \frac{1}{\sqrt{2}}\sin(\sqrt{2}x).$$

2. $x^2y'' + 5xy' + (4+\pi^2)y = \ln x, \;\; y(1) = 0, y(e) = 0.$
   With the change of variables $x = e^t$, the ODE can be written as

   $$y'' + 4y' + (4+\pi^2)y = t,$$

   with the corresponding initial conditions $y(0) = 0$ and $y(1) = 0$. The general solution of this ODE is

   $$y(t) = c_1e^{-2t}\cos(\pi t) + c_2e^{-2t}\sin(\pi t) + \frac{t\pi^2 + 4t - 4}{(4+\pi^2)^2}.$$

Imposing the boundary conditions, it is necessary that

$$c_1 - \frac{4}{(4+\pi^2)^2} = 0,$$

and

$$-e^{-2}c_1 + \frac{\pi^2}{(4+\pi^2)^2} = 0.$$

Hence no solution exists.

3. Find the eigenvalues and eigenfunctions of the given boundary value problem. Assume that all eigenvalues are real,

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(L) = 0.$$

Assuming that $\lambda > 0$. Setting $\lambda = \mu^2$, the general solution of the resulting ODE is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$, with $y'(x) = -\mu c_1 \sin(\mu x) + \mu c_2 \cos(\mu x)$. Imposing the first boundary condition, we find that $c_2 = 0$. Therefore, $y(x) = c_1 \cos(\mu x)$. The second boundary condition gives $c_1 \cos(\mu L) = 0$. For a nontrivial solution, it is necessary that $\cos(\mu L) = 0$, that is $\mu = (2n-1)\pi/(2L)$, with $n = 1, 2, \ldots$ Therefore the eigenvalues are

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}, n = 1, 2, \ldots.$$

The corresponding eigenfunctions are given by

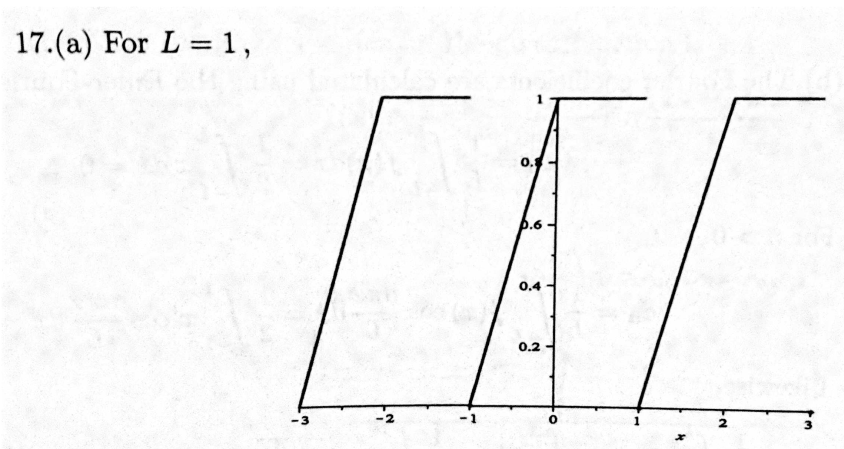
$$y_n = \cos\frac{(2n-1)\pi x}{2L}, n = 1, 2, \ldots.$$

Part 3: Fourier Series

1. Sketch the graph of the function below for three periods. Then, find the Fourier series for the given function.

$$f(x) = \begin{cases} x + L, & -L \le x \le 0, \\ L, & 0 \le x \le L; \end{cases} \qquad f(x + 2L) = f(x).$$

Here is a graph of $f(x)$:



The Fourier coefficients are calculated using the Euler-Fourier formulas:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_{-L}^{0}(x+L)dx + \frac{1}{L}\int_{0}^{L} Ldx = 3L/2.$$

For $n > 0$,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx =$$

$$= \frac{1}{L}\int_{-L}^{0}(x+L)\cos\left(\frac{n\pi x}{L}\right)dx + \frac{1}{L}\int_{0}^{L} L\cos\left(\frac{n\pi x}{L}\right)dx = \frac{L(1-\cos(n\pi))}{n^2\pi^2}.$$

Likewise,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx =$$

$$= \frac{1}{L}\int_{-L}^{0}(x+L)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{0}^{L} L\sin\left(\frac{n\pi x}{L}\right)dx = \frac{-L\cos(n\pi)}{n\pi}.$$

Note that $\cos(n\pi) = (-1)^n$. It follows that the Fourier series for the given function is

$$f(x) = \frac{3L}{4} + \frac{L}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{2}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{L} - \frac{(-1)^n\pi}{n}\sin\frac{n\pi x}{L}\right].$$