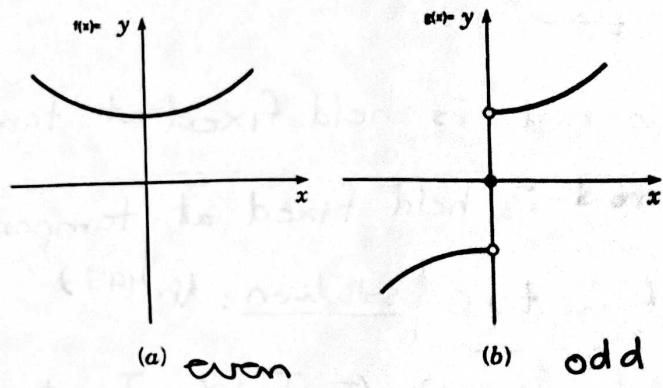


### Chapters 10.4-10.6 Review

**Objectives in class:** (1) Review properties of odd and even functions (2) Practice the method of separation of variables to solve the heat equation with various boundary conditions. **Objectives at home:** (1) Study the lecture notes posted on Blackboard covering the derivation of the Heat Equation (2) carefully read Chapters 10.5-10.6.

#### Part 1: Odd and Even Functions

Consider the two functions graphed below:



what about  
odd+odd, even+even<sup>2</sup>.

- Determine whether the following functions are odd, even, or neither:

- $f(x) + g(x)$  even + odd = neither,  $f(-x) + g(-x) = f(x) - g(x)$ .
- $f(x)g(x)$  (even)(odd) = odd,  $f(-x)g(-x) = -f(x)g(x)$
- $f^2(x)$  (even)(even) = even,  $f(-x)f(-x) = f(x) \cdot f(x)$
- $g^2(x)$  (odd)(odd) = even,  $g(-x)g(-x) = (-1)(-1) g(x)g(x) = g^2(x)$ .

- Which function can be represented by a Fourier series of the form:

odd functions, so  
 $g(x)$

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

- Which function can be represented by a Fourier series of the form:

even functions, so  
 $f(x)$ .

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

- Are the following functions odd, even, or neither
- $\tan(2x)$  odd,  $\tan(2x) = \frac{\sin(2x)}{\cos(2x)}$ ,  $\frac{\text{odd}}{\text{even}} = \text{odd}$ .
  - $\sec x$  even,  $\sec x = \frac{1}{\cos x} = \frac{\text{even}}{\text{even}} = \text{even}$
  - $e^{-x}$  neither

#### Part 2: Separation of Variables; Heat Equation

Consider a uniform bar of length  $L$  having an initial temperature distribution given by  $f(x)$ ,  $0 \leq x \leq L$ . Write down the PDE modeling the following scenarios, and use the method of separation of variables to write down a the general solution for each.

1. Assume that the temperature of the bar is fixed at zero at the end points.

$$\alpha^2 u_{xx} = u_t, \quad 0 < x < L, \quad t > 0 \quad | \quad \text{Solution: (P. 491-492)}$$

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-n^2 \pi^2 \alpha^2 t / L^2} \sin\left(\frac{n\pi x}{L}\right)$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq L$$

$$u(0, t) = u(L, t) = 0, \quad t > 0.$$

$$| \quad c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

The steady state solution  $u^*(x) = \lim_{t \rightarrow \infty} u(x, t) = 0$ .

2. Assume that the left end of the rod is held fixed at temperature  $T_1$  and the right end of the rod is held fixed at temperature  $T_2$ .

$$\alpha^2 u_{xx} = u_t, \quad 0 < x < L, \quad t > 0 \quad | \quad \text{solution: (P. 497)}$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq L \quad | \quad u(x, t) = (T_2 - T_1) \frac{x}{L} + T_1 +$$

$$| \quad \sum_{n=1}^{\infty} c_n e^{-n^2 \pi^2 \alpha^2 t / L^2} \sin\left(\frac{n\pi x}{L}\right)$$

The steady state solution is

$$| \quad c_n = \frac{2}{L} \int_0^L (f(x) - (T_2 - T_1) \frac{x}{L} - T_1) \sin\left(\frac{n\pi x}{L}\right) dx.$$

$$U(x) = \lim_{t \rightarrow \infty} u(x, t) = (T_2 - T_1) \frac{x}{L} + T_1$$

3. Assume that the ends of the bar are insulated so that there's no passage of heat through them.

(P. 499-500)

$$\alpha^2 u_{xx} = u_t, \quad 0 < x < L, \quad t > 0 \quad | \quad \text{solution:}$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq L \quad | \quad u(x, t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2 \pi^2 \alpha^2 t / L^2} \cos\left(\frac{n\pi x}{L}\right)$$

$$u_x(0, t) = 0, \quad u_x(L, t) = 0, \quad t > 0 \quad | \quad c_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 0, 1, 2, \dots$$

The steady state solution is

$$\frac{c_0}{2} = \frac{1}{L} \int_0^L f(x) dx \quad \text{which is the average of } f(x) \text{ on the interval } 0 < x < L.$$

Part 3 #1 By part 2 #1, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2 \pi^2 100t/40^2} \sin\left(\frac{n\pi x}{40}\right)$$

where  $C_n = \frac{2}{40} \int_0^{40} f(x) \sin\left(\frac{n\pi x}{40}\right) dx$

orthogonality  
see \*  
below

$$\begin{aligned} C_n &= \frac{2}{40} \int_0^{40} f(x) \sin\left(\frac{n\pi x}{40}\right) dx \\ &= \frac{2}{40} \left[ \int_0^{40} \sin(2\pi x) \sin\left(\frac{n\pi x}{40}\right) dx + \frac{2}{40} \int_0^{40} \sin(3\pi x) \sin\left(\frac{n\pi x}{40}\right) dx \right] \\ &= 1 \text{ if } n=80 \text{ or } m=120 \text{ and zero otherwise.} \end{aligned}$$

$$\text{So, } u(x,t) = 80 C e^{-(800\pi)^2 t/40^2} \sin(2\pi x) + 120 e^{-(1200\pi)^2 t/40^2} \sin(3\pi x). \text{ The steady}$$

state is  $U(x) = 0$ .

Part 3 #2 By part 2 #2, the solution is:

$$u(x,t) = \frac{20x}{40} + 10 + \sum_{n=1}^{\infty} C_n e^{-n^2 \pi^2 10^2 t/40^2} \sin\left(\frac{n\pi x}{40}\right)$$

$$\text{where } C_n = \frac{1}{20} \int_0^{40} \left[ \sin(2\pi x) + \frac{x}{2} \right] - \left( \frac{x}{2} + 30 \right) \sin\left(\frac{n\pi x}{40}\right) dx$$

$$\begin{aligned} &= \frac{1}{20} \int_0^{40} \sin(2\pi x) \sin\left(\frac{n\pi x}{40}\right) dx - \frac{10}{20} \int_0^{40} \sin\left(\frac{n\pi x}{40}\right) dx \\ &\quad \text{* use orthogonality} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{20} \int_0^{40} \sin(2\pi x) \sin\left(\frac{n\pi x}{40}\right) dx + \frac{1}{2} \left( \cos\left(\frac{n\pi x}{40}\right) \cdot \frac{40}{n\pi} \right) \Big|_0^{40} \\ &\quad \text{i.e. } \begin{cases} \sin(n\pi x) \\ \sin(n\pi x) dx \end{cases} = \frac{1}{20} \int_0^{40} \sin(2\pi x) \sin\left(\frac{n\pi x}{40}\right) dx + \frac{1}{20} \left[ \frac{(-1)^n}{n\pi} - \frac{1}{n\pi} \right] \Big|_0^{40} \\ &\quad \text{if } n=0 \\ &\quad \text{if } n \neq 0 \end{aligned}$$

If  $n = 80$ ,

$$a_{80} = 1$$

If  $n \neq 80$  &  $n$  is even then

$$a_n = 0.$$

If  $n$  is odd then

$$a_n = \frac{40}{n\pi} = -\frac{40}{n\pi}$$

$$\text{So, } u(x,t) = \frac{x}{2} + 80 + e^{-[80\pi t/40]^2} \sin(2\pi x) + \sum_{k=0}^{\infty} \frac{1}{(2k+1)} e^{-(2k+1)^2 \pi^2 t/40^2} \sin\left(\frac{(2k+1)\pi x}{40}\right).$$

The steady state solution is  $U(x) = \frac{x}{2} + 30$ .

part 3 #3] By part 2 #3, we have

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2 \pi^2 t/40^2} \cos\left(\frac{n\pi x}{40}\right) \text{ with}$$

$$c_n = \frac{2}{40} \int_0^{40} \cos(C\pi x) \cdot \cos\left(\frac{n\pi x}{40}\right) dx = 1 \text{ if } n=240 \quad \text{and}$$

$$0, u(x,t) = e^{-240^2 \pi^2 t/40^2} \cos(6\pi x). \quad = 0 \text{ if } n \neq 240 \quad \text{otherwise.}$$

The steady state is  $\frac{c_0}{2} = 0$  in this case. The steady state is the average of the initial temp. distribution. This makes sense b/c the ends are equal to the average.