Generalization of the Calderón-Zygmund Decomposition

Cornell University Math: 6120 Final Essay

Aleksandra Niepla

May 23, 2016

1 Introduction

The Calderón-Zygmund decomposition, named after Alberto Calderón and his advisor Antoni Zygmund, is a key result in the analysis of singular integrals, Fourier analysis, and harmonic analysis. The idea of this decomposition is to break up an arbitrary integrable function into a "good" part and a "bad" part, and use different techniques to analyze each of them.

Specifically, if we are given a function $f : \mathbb{R}^d \to \mathbb{C}$ and an altitude α, we write $f = g + b$, where $|g|$ is pointwise bounded by a constant multiple of α. Moreover b can be large, but it has two useful properties: it is supported in a set of small measure, and its mean value is zero on each of the balls that make up its support.

All the proofs and constructions of this essay are taken from [St]. The goal is to summarize the main ideas and theorems from the first few sections of the text, regarding the Calderón-Zygmund decomposition.

2 Preliminaries

2.1 Basic Assumptions

To begin, we consider the collection of Euclidean balls $\{B(x,\delta)\}_\delta$ in $\mathbb{R}^n$. Where, as usual we take $B = B(x,\delta)$ to be the ball, centered at x of radius δ. Then, taking μ to be the Euclidean measure, we see that this family of balls satisfies the following: There exist constants c_1 and c_2, both greater than 1, so that, for all x, y, and δ,

(i) $B(x,\delta) \cap B(y,\delta) \neq \emptyset$ implies $B(y,\delta) \subset B(x, c_1\delta)$.

(ii) $\mu(B(x, c_1\delta)) \leq c_2\mu(B(x,\delta))$.

(iii) $\cap_\delta \overline{B}(x,\delta) = \{x\}$ and $\cup_\delta B(x,\delta) = \mathbb{R}^n$.

(iv) For each open set U and each $\delta > 0$, the function $x \to \mu(\{B(x,\delta) \cap U\})$ is continuous.

Statement (i) guarantees the engulfing property that is needed to prove the Vitali-type covering lemmas seen in the next section, while assumption (ii) represents the fact that μ is a "doubling" measure. An equivalent way of stating property (ii) is with the inequality $\mu(B(x, 2\delta)) \leq c_2'\mu(B(x,\delta))$, from which the terminology "doubling" originates.

In what follows, we generalize the collection above. That is, for each $x \in \mathbb{R}^n$, let $\{B(x,\delta)\}_\delta$ be a collection of nonempty, open, bounded subsets of $\mathbb{R}^n$, parametrized by $\delta, 0 < \delta < \infty$. We suppose that these generalized "balls" are monotonic in δ in the sense that $B(x,\delta_1) \subset B(x,\delta_2)$ whenever $\delta_1 < \delta_2$. We also assume that we are given a nonnegative Borel measure μ with the property that $\mu(\mathbb{R}^n) > 0$. Finally, we take the properties above as postulates. To see examples of such collections (other than Euclidean balls with the standard Euclidean measure) see p. 9 of [St].

2.2 Additional Definitions

It follows from the properties above that for any locally integrable f, and any $\delta > 0$, the mean value

$$(A_\delta)f(x) = \frac{1}{\mu(B(x,\delta))} \int_{B(x,\delta)} f(y)d\mu(y)$$

is a continuous function of x.

In addition, we define the *maximal function*, by

$$(Mf)(x) = \sup_{\delta > 0} A_\delta(|f|)(x).$$

We also defined the larger "uncentered" maximal function $\widetilde{M}f$ as follows

$$(\widetilde{M}f)(x) = \sup_{x \in B} \frac{1}{\mu(B)} \int_B |f(y)|d\mu(y).$$

Moreover, we have that $(Mf)(x) \leq (\widetilde{M}f)(x)$.

3 Covering lemmas

In this section, we show the reasoning for the postulates chosen in the previous section by introducing Vitali and Whitney type covering lemmas, along with some of their consequences. First, we consider the simplest of these covering lemmas, a finite version of the Vitali lemma.

Lemma 3.0.1 *Let E be a measurable subset of $\mathbb{R}^n$ that is the union of a finite collection of balls $\{B_j\}$. Then one can select a disjoint subcollection $B_1, \ldots, B_m$ of the $\{B_j\}$ so that*

$$\sum_{k=1}^{m} \mu(B_k) \geq c\mu(E).$$

Here c is a positive constant.

Proof. Can be proven using postulates (i) and (ii) above. A full proof can be found on p. 12 of [St.]

□

The lemma just stated allows us to obtain some fundamental results about averages $A_\delta(f)$ and the maximal function $M(f) = \sup_\delta A_\delta(|f|)$.

Theorem 3.0.2 *Let f be a function defined on $\mathbb{R}^n$.*

(a) If $f \in L^p, 1 \leq p \leq \infty$, then $M(f)$ is finite almost everywhere.

(b) If $f \in L^1$, then for every $\alpha > 0$,

$$\mu(\{x : (Mf)(x) > \alpha\}) \leq \frac{c_2}{\alpha} \int_{\mathbb{R}^n} |f(y)| d\mu(y)$$

(c) If $f \in L^p, 1 \leq p \leq \infty$, then $M(f) \in L^p$ and

$$||M(f)||_p \leq A_p ||f||_p$$

where the bound A_p depends only on c_2 and p.

Proof. See p. 13. Note that the proof shows the inequalities of the theorem hold for the larger "uncentered" function $\widetilde{M}f$. We will use this fact in the next section.

□

Corollary 3.0.3 *If f is locally integrable with respect to $d\mu$, then*

$$lim_{\delta \to 0}(A_\delta f)(x) = f(x)$$

for almost every x.

In order to prove the generalized Calderón-Zygmund decomposition, we will need a stronger covering lemma. In the lemma that follows the idea is that if we are given a nonempty closed set F, then we can "cover" F^c by a collection of balls that are almost disjoint, and whose sizes are comparable to their distances from the set F.

As pointed out in [St], in the standard setting of $\mathbb{R}^n$, the covering of F^c can be done by closed cubes whose interiors are disjoint and whose side lengths are comparable to their distances from the set F.

For the general setting it is more complicated. We start by fixing a pair of positive constants c^* and c^{**} (with $1 \leq c^* \leq c^{**}$), which will depend only the quantity c_1 appearing in assumption (i), and not the particular set F in question. Using them, for any ball B we define the balls B^* and B^{**} that have the same "centers" as B but whose "radii" are expanded by the factors c^* and c^{**} respectively. More precisely, if $B = B(x, \delta)$ then we define $B^* = B(x, c^*\delta)$ and $B^{**} = B(x, c^{**}\delta)$.

Lemma 3.0.4 *Given F, a closed nonempty set, there exists a collection of balls $B_1, \ldots, B_k, \ldots$ so that*

(a) The B_k are pairwise disjoint.

(b) $\bigcup_k B_k^* = O = F^c$

(c) $B_k^{**} \cap F \neq \emptyset$, for each k.

Proof. Denote $F^c = O$. Following the proof of Stein, we start by choosing ϵ small enough; later it will be determined that $\epsilon = \frac{1}{8c_1^2}$ will work. With ϵ fixed, consider the covering $\{B(x, \epsilon\delta(x))\}_{x \in O}$ of O, where $\delta(x)$ is the "distance" of x from F, namely, $\delta(x) = sup\{\delta : B(x, \delta) \subset O\}$. By postulate (iii), we have that for each $x \in O$ the function $\delta(x)$ is strictly positive and finite.

Next, choose a maximal disjoint subcollection of $\{B(x, \epsilon\delta(x)))\}_{x \in O}$; for this subcollection, consider $B_1, \ldots, B_k, \ldots$ with $B_k = B(x_k, \epsilon\delta(x_k))$, we will prove (a), (b), and (c) for this choice of $\{B_k\}_k$. Set $c^* = \frac{1}{2\epsilon}, c^{**} = \frac{2}{\epsilon}$. Then, we have
$B_k^* = B(x_k, \delta(x_k)/2)$, and $B_k^{**} = B(x_k, 2\delta(x_k))$,

Notice that (a) and (c) hold automatically by the choice of B_k. It is also clear that $B_k^* \subset O$; so, what is left is to show that $O \subset \bigcup_k B_k^*$.

Now let $x \in O$; then, by the maximality of the collection $\{B_k\}$,

$$B(x_k, \epsilon\delta(x_k)) \cap B(x, \epsilon\delta(x)) \neq \emptyset$$

for some k.

Then, $\delta(x_k) \geq \frac{\delta(x)}{4c_1}$. Since, if not, taking $\epsilon < \frac{1}{2c_1}(< 1)$, we have

$$B(x_k, \delta(x_k)) \cap B(x, \frac{\delta x}{2c_1}) \neq \emptyset.$$

Since $2\delta(x_k) < \frac{\delta(x)}{2c_1}$, by the engulfing property

$$B(x_k, 2\delta(x_k)) \subset B(x, \frac{\delta(x)}{2}),$$

which gives a contradiction since $B(x_k, 2\delta(x_k))$ intersects $F = O^c$, while $B(x, \frac{\delta(x)}{2} \subset O$.

Using $4c_1\epsilon\delta(x_k) \geq \epsilon\delta(x)$ and the engulfing property again gives

$$x \in B(x_k, c_1 4c_1\epsilon\delta(x_k)).$$

We take $B(x_k, c_1 4c_1\epsilon\delta(x_k)) = B_k^* = B(x_k, \delta(x_k)/2)$; i.e., $c^* = 4c_1^2$, $\epsilon = \frac{1}{2c^*} = \frac{1}{8c_1^2}$, $c^{**} = 4c^* = 16c_1^2$, finishing the proof.

$\square$

4 The Generalized Calderón-Zygmund Decomposition

Finally, we are ready to make precise the generalized Calderón-Zygmund decomposition that was discussed in the introduction.

Theorem 4.0.1 *Suppose we are given a function $f \in L^1$ and a positive number α, with*

$$\alpha > \frac{1}{\mu(\mathbb{R}^n)} \int_{\mathbb{R}^n} |f| d\mu.$$

Then there exists a decomposition of f, $f = g + b$, with $b = \sum_k b_k$, and a sequence of balls $\{B_k^*\}$, so that

1. *$|g(x)| \leq c\alpha$, for a.e. x*

2. *Each b_k is supported in B_k^*,*

$$\int |b_k(x)| d\mu(x) \leq c\alpha\mu(B_k^*), \text{and} \int b_k(x) d\mu(x) = 0.$$

3. $\sum_k \mu(B_k^*) \leq \frac{c}{\alpha} \int |f(x)| d\mu(x)$.

Proof. Let $E_\alpha = \{x : \widetilde{M} f(x) > \alpha\}$, where $\widetilde{M}$ is the uncentered maximal function defined in section 2. E_α is an open set, we will start by considering the case when its complement is nonempty.

We can apply the lemma 3.0.2 to $O = E_\alpha$. So, we obtain collection of balls $\{B_k\}, \{B_k^*\}$. Moreover, we will leave it as an exercise that we can construct a collection of "cubes" $\{Q_k\}$, such that the Q_k are disjoint, their union is O, and $B_k \subset Q_k \subset B_k^*$. It follows that

$$\sum_k \mu(B_k) \leq \mu(E_\alpha). \tag{1}$$

Now define $g(x) = f(x)$ for $x \notin E_\alpha$, and

$$g(x) = \frac{1}{\mu(Q_k)} \int_{Q_k} f(y) d\mu(y),$$

if $x \in Q_k$.

Hence $f = g + \sum b_k$, where

$$b_k(x) = \chi_{Q_k} [f(x) - \frac{1}{\mu(Q_k)} \int_{Q_k} f(y) d\mu(y)], \tag{2}$$

with χ_{Q_k} denoting the characteristic function of Q_k.

From the differentiation theorem, we have $|f(x)| \leq \alpha$ for a.e. $x \in (\bigcup Q_k)^c = \{x : \widetilde{M} f(x) \leq \alpha\}$. So $|g(x)| \leq \alpha$ for $x \in (\bigcup Q_k)^c$. Next we observe that

$$\frac{1}{\mu(B_k^{**})} \int_{B_k^{**}} |f(x)| d\mu(x) \leq \alpha \tag{3}$$

because the ball B_k^{**} intersects E_α^c.

So from 3, and the fact that $B_k \subset Q_k \subset B_k^{**}$. Also,

$$\int |b_k(x)| d\mu(x) \leq 2 \int_{Q_k} |f(x)| d\mu(x) \leq c\alpha\mu(B_k^*)$$

by 3 and the doubling property. Moreover, that $\int b_k(x) d\mu(x) = 0$ is obvious from 2. Hence the second part of the theorem is proved. Again by the doubling property, $\sum \mu(B_k^*) \leq c\mu(\{\tilde{M} f > \alpha\})$ because of 1, and the quantity on the right is dominated by $(c/\alpha) \int |f| d\mu$, as we see using theorem 3.0.2. So, we have proven the theorem in the case that $\{x : \tilde{M} f(f) \leq \alpha\} \neq \emptyset$.

Finally, we now consider the remaining case when $\{x : \widetilde{M} f(x) > \alpha\} = \mathbb{R}^n$ (which can happen only when $\mu(\mathbb{R})^n) < \infty$), then we see by theorem 3.0.2 that

$$\mu(\mathbb{R})^n \leq \frac{c}{\alpha} \int_{(\mathbb{R})^n} |f| d\mu.$$

So, we get the decomposition $f = g + b_1$, with

$$g = \frac{1}{\mu(\mathbb{R})^n} \int_{(\mathbb{R})^n} f d\mu,$$

$b_1 = f - g$; here b_1 is supported in the "ball" $B_1^* = \mathbb{R}^n$. Our assumption that

$$\alpha > \frac{1}{\mu(\mathbb{R})^n} \int_{(\mathbb{R})^n} |f| d\mu$$

guarantees that $|g| \leq \alpha$.

$\square$

5 Application

In this section we describe, without much detail, a standard application of the Calderón-Zygmund decomposition. First we state from [St] a weaker version of the Riesz-Thorin interpolation theorem covered in class. This theorem is called the Marcinkiewicz interpolation theorem, and it is formulated as follows:

Theorem 5.0.1 *Let T be a bounded linear operator from L^p to $L^{p,w}$ and at the same time from L^q to $L^{q,w}$. Then T is also a bounded operator from L^r to L^r.*

This theorem is often used to extend operators, such as the fractional integral operators defined below.

Definition 5.0.2 When $0 < n < d$, we define the fractional integral operator F_α for $0 < \alpha < n$ by

$$F_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{(n-\alpha)}} d\mu(y), x \in \mathbb{R}^d.$$

It turns out that the hypothesis of the Marcinkiewicz interpolation theorem is easy to show in the case of fractional integral operators. However, if we consider the case when $\alpha = 0$ in the definition above, we get what is called a singular integral operator. Singular integrals were also introduced by Calderón and Zygmund, and an extensive discussion of them can be found in [St]. However, in this case the hypothesis of the Marcinkiewicz interpolation theorem is not as easy to show. In fact, the hypothesis is established by a direct application of the Calderón-Zygmund decomposition, and hence allows us to extend these operators as well. The detailed proof of this can be found on p. 20 of [St].

References

[St] Stein, E., *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press: New Jersey, 1993.