p.10: show that problem 10 has a unique solution;
p.31: #1-5

Problem 10: The initial value problem is $u_x + u_y + u = e^{x+2y}$ with $u(x,0)=0$. Assume that $u^1(x,y)$ and $u^2(x,y)$ are solutions. Consider $v(x,y) := u^1(x,y) - u^2(x,y)$. Then,

$$
\begin{aligned}
v_x + v_y + v &= u_x^1 - u_x^2 + u_y^1 - u_y^2 + u^1 - u^2 \\
&= (u_x^1 + u_y^1 + u^1) - (u_x^2 + u_y^2 + u^2) \\
&= e^{x+2y} - e^{x+2y} \\
&= 0,
\end{aligned}
\quad \text{and} \quad
\begin{aligned}
v(x,0) &= u^1(x,0) - u^2(x,0) \\
&= 0 - 0 \\
&= 0.
\end{aligned}
$$

Hence, $v(x,y)$ satisfies the initial value problem:

(*) $v_x + v_y + v = 0$, $v(x,0) = 0$. We can solve this by reducing it to an ODE via the coordinate change: $x' = x+y$, $y' = x-y$. Then, as on p. 7, by the chain rule:

$V_x = V_{x'} + V_{y'}$ and $V_y = V_{x'} - V_{y'}$. Plugging into (*):

$2V_{x'} + V = 0$. This is a linear 1st order ODE w/ general solution: $v(x',y') = e^{-1/2x'} f(y') \Leftrightarrow v(x,y) = e^{-\frac{(x+y)}{2}} f(x-y)$. Since $v(x,0) = e^{-x/2} f(x) = 0 \Rightarrow f(w) = 0$ for all w. Hence, $f(x-y) = 0$ for all x,y. $\Rightarrow v(x,y) = u^1(x,y) - u^2(x,y) \equiv 0$. So, $u^1 = u^2$. The solution is unique.

Problem 1: What is the type of each of the following equations?

(a) $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0$

(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$

Solution: (a) using that $u_{xy} = u_{yx}$ the equation is equivalent to $u_{xx} - 4_{xy} + u_{yy} + 2u_y + 4u = 0$

$$a_{11} = 1, \quad a_{22} = 1, \quad 2a_{12} = -4 \quad \Rightarrow \quad a_{12}^2 > a_{11}a_{22}$$

so the equation is $\boxed{\text{Hyperbolic}}$.

(b)

$$a_{11} = 9, \quad a_{22} = 1, \quad 2a_{12} = 6 \quad \Rightarrow \quad a_{12}^2 = a_{11}a_{22}$$

so the equation is $\boxed{\text{Parabolic}}$

Problem 2: Find the regions in the xy plane where the equation $(1+x)u_{xx} + 2xy\,u_{xy} - y^2 u_{yy} = 0$ is elliptic, hyperbolic, or parabolic. Sketch them.

Solution: For this PDE we have $a_{11} = 1+x$, $a_{12} = xy$, $a_{22} = -y^2$.

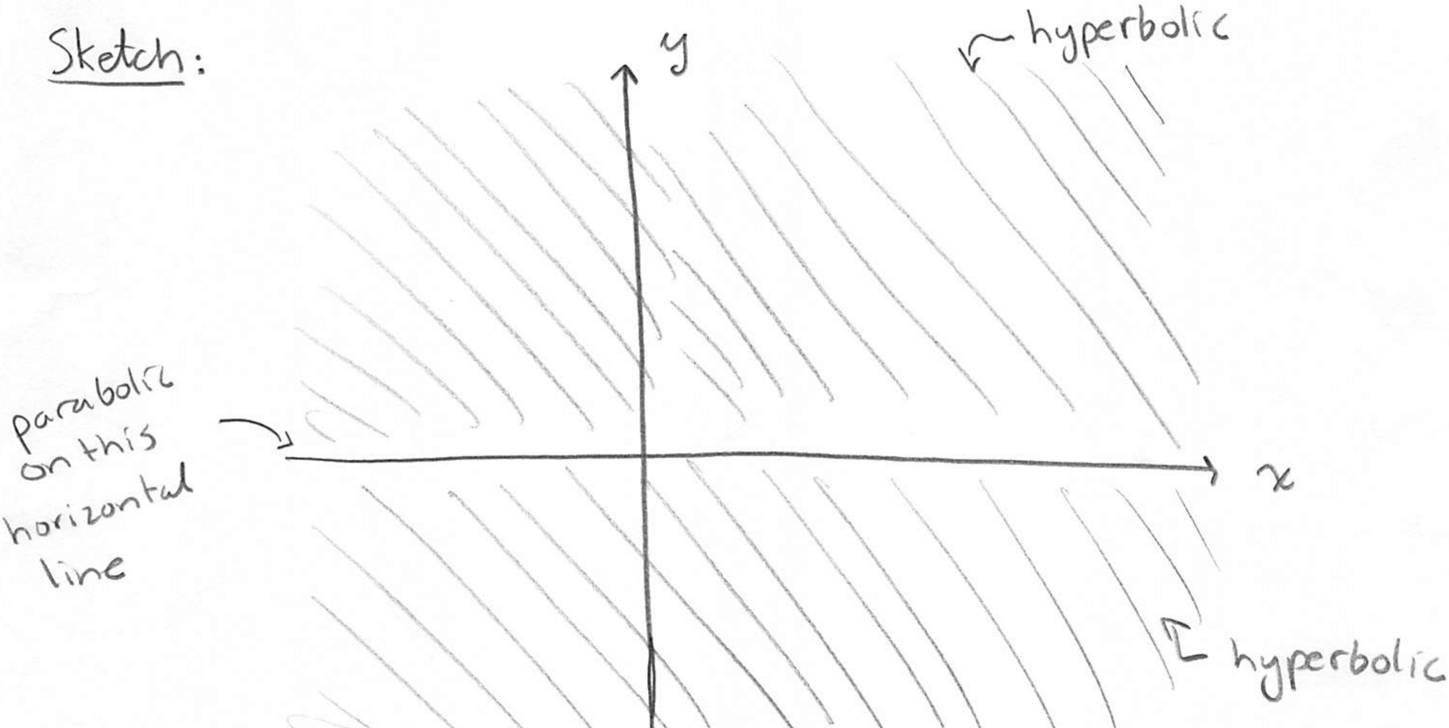
We want to study the sign of

$$a_{12}^2 - a_{11}a_{22} = x^2y^2 - (1+x)(-y^2) = y^2(x^2+x+1).$$

Since $y^2 \geq 0$, it remains to check the sign of x^2+x+1. This is an upward opening parabola whose min. occurs when $2x+1=0 \Rightarrow$ min. is at $x=-1/2$. Moreover $\left(-\frac{1}{2}\right)^2 - \frac{1}{2} + 1 = \frac{1}{4} - \frac{1}{2} + 1 = 3/4 > 0$. So, $x^2+x+1 > 0$ for all x.

So, PDE is never elliptic. It is parabolic only for $y=0$, and hyperbolic otherwise.

Sketch:



Problem 3: Among all the equations of the form (1), show that the only ones that are unchanged under all rotations (rotationally invariant) have the form $a(u_{xx} + u_{yy}) + bu = 0$.

Solution: Fix a vector $(x, y) \in \mathbb{R}^2$, a counterclockwise rotation by an angle θ is given by

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix},$$

so the equations that are rotationally invariant are the ones invariant under the coordinate change

$x' = x\cos\theta - y\sin\theta$

$y' = x\sin\theta + y\cos\theta$, using the multivariable chain rule we compute:

$$u_x = \frac{\partial u}{\partial x'} \cdot \frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'} \frac{\partial y'}{\partial x} = u_{x'} \cdot \cos\theta + u_{y'} \sin\theta$$

$$u_y = \frac{\partial u}{\partial x'} \cdot \frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'} \cdot \frac{\partial y'}{\partial y} = -u_{x'} \sin\theta + u_{y'} \cos\theta$$

$$u_{xx} = \cos\theta \left[\frac{\partial^2 u}{\partial x'^2} \cdot \frac{\partial x'}{\partial x} + \frac{\partial^2 u}{\partial y' \partial x'} \cdot \frac{\partial y'}{\partial x} \right] + \sin\theta \left[\frac{\partial^2 u}{\partial x' \partial y'} \cdot \frac{\partial x'}{\partial x} + \frac{\partial^2 u}{\partial y'^2} \cdot \frac{\partial y'}{\partial x'} \right]$$

$$= \cos\theta \left[u_{x'x'} \cdot \cos\theta + u_{y'x'} \sin\theta \right] + \sin\theta \left[u_{x'y'} \cos\theta + u_{y'y'} \cdot \sin\theta \right]$$

$$= \cos^2\theta \, u_{x'x'} + \sin^2\theta \, u_{y'y'} + 2\cos\theta\sin\theta \, u_{x'y'},$$

$$u_{yy} = -\sin\theta\left[\frac{\partial^2 u}{\partial x'^2}\cdot\frac{\partial x'}{\partial y} + \frac{\partial^2 u}{\partial y'\partial x'}\cdot\frac{\partial y'}{\partial y}\right] + \cos\theta\left[\frac{\partial u}{\partial x'\partial y'}\cdot\frac{\partial x'}{\partial y} + \frac{\partial^2 u}{\partial y'^2}\cdot\frac{\partial y'}{\partial y}\right]$$

$$= -\sin\theta\left[-u_{x'x'}\cdot\sin\theta + u_{y'x'}\cos\theta\right] + \cos\theta\left[-u_{x'y'}\sin\theta + u_{y'y'}\cos\theta\right]$$

$$= \sin^2\theta\, u_{x'x'} + \cos^2\theta\, u_{y'y'} - 2\sin\theta\cos\theta\, u_{x'y'}$$

$$u_{xy} = -\sin\theta\left[\frac{\partial^2 u}{\partial x'^2}\cdot\frac{\partial x'}{\partial x} + \frac{\partial^2 u}{\partial y'\partial x'}\cdot\frac{\partial y'}{\partial x}\right] + \cos\theta\left[\frac{\partial^2 u}{\partial x'\partial y'}\cdot\frac{\partial x'}{\partial x} + \frac{\partial^2 u}{\partial y'^2}\cdot\frac{\partial y'}{\partial x}\right]$$

$$= -\sin\theta\left[u_{x'x'}\cdot\cos\theta + u_{y'x'}\sin\theta\right] + \cos\theta\left[u_{x'y'}\cos\theta + u_{y'y'}\sin\theta\right]$$

$$= \cos\theta\sin\theta\, u_{y'y'} - \cos\theta\sin\theta\, u_{x'x'} + (\cos^2\theta - \sin^2\theta)\, u_{x'y'}.$$

<u>Plugging into (1) we get:</u>* $a_{11}\cos^2\theta\, u_{x'x'} + a_{11}\sin^2\theta\, u_{y'y'} + a_{11}2\cos\theta\sin\theta\, u_{x'y'}$

$+2a_{12}\cos\theta\sin\theta\, u_{y'y'} - 2a_{12}\cos\theta\sin\theta\, u_{x'x'} + 2a_{12}(\cos^2\theta - \sin^2\theta)u_{x'y'} +$

$a_{22}\sin^2\theta\, u_{x'x'} + a_{22}\cos^2\theta\, u_{y'y'} - 2a_{22}\sin\theta\cos\theta\, u_{x'y'} + a_1\cos\theta\, u_{x'}$

$+a_1\sin\theta\, u_{y'} + a_2\cos\theta\, u_{y'} - a_2\sin\theta\, u_{x'} + a_0 u = 0.$

Grouping like terms and using the fact that we want rotational invariance:

$$\begin{cases} a_1\cos\theta - a_2\sin\theta = a_1 & (1)\\ a_1\sin\theta + a_2\cos\theta = a_2 & (2)\\ a_{11}\cos^2\theta + a_{22}\sin^2\theta - 2a_{12}\cos\theta\sin\theta = a_{11} & (3)\\ a_{11}\sin^2\theta + a_{22}\cos^2\theta + 2a_{12}\cos\theta\sin\theta = a_{22} & (4)\\ a_{11}\cos\theta\sin\theta + a_{12}(\cos^2\theta - \sin^2\theta) - a_{22}\sin\theta\cos\theta = a_{12} & (5)\end{cases}$$

must hold for <u>all</u> θ.

plug $\theta = \frac{\pi}{2}$ into (1) and (2) $\Rightarrow \begin{cases} -a_2 = a_1 \\ a_1 = a_2 \end{cases} \Rightarrow a_1 = a_2 = 0.$

plug $\theta = \frac{\pi}{2}$ into (3) $\Rightarrow \{ a_{22} = a_{11}$

using that $a_{22} = a_{11}$ and rearranging (5) we obtain $a_{12}[\cos^2\theta - \sin^2\theta - 1] = 0$,

taking $\theta = \pi/6 \Rightarrow a_{12}\left[\frac{3}{4} - \frac{1}{4} - 1\right] = -\frac{a_{12}}{2} = 0 \Rightarrow a_{12} = 0.$

Hence $a_1 = a_2 = a_{12} = 0$ and $a_{22} = a_{11}$ are necessary conditions for rotational invariands, to see they are sufficient, plug them back into ✱: $a_{11}(u_{x'x'} + u_{y'y'}) + a_0 u = 0$

Plugging them into (1): $a_{11}(u_{xx} + u_{yy}) + a_0 u = 0$ ✓

Problem 4: What is the type of the equation

$u_{xx} - 4u_{xy} + 4u_{yy} = 0$? Show by direct substitution

that $u(x,y) = f(y+2x) + xg(y+2x)$ is a solution for arbitrary differentiable functions f and g.

Solution:

$$\left.\begin{array}{l} a_{11} = 1 \\ a_{12} = -2 \\ a_{22} = 4 \end{array}\right. \Rightarrow a_{12}^2 - 4 = 0 \Rightarrow \boxed{\text{PDE is parabolic}}$$

To verify that $u(x,y)$ is a solution we compute the relevant derivatives:

$$u_x = 2f'(y+2x) + g(y+2x) + 2xg'(y+2x)$$

$$\underline{u_{xx}} = 4f'(y+2x) + 2g'(y+2x) + 2g'(y+2x) + 4xg''(y+2x)$$

$$u_y = f'(y+2x) + xg'(y+2x)$$

$$\underline{u_{yy}} = f''(y+2x) + xg''(y+2x)$$

$$\underline{u_{xy}} = 2f''(y+2x) + g'(y+2x) + 2xg''(y+2x) \quad \text{So,}$$

$$\begin{aligned} u_{xx} - 4u_{xy} + 4u_{yy} &= 4f'(y+2x) + 4g'(y+2x) + 4xg''(y+2x) \\ &\quad - 8f''(y+2x) - 4g'(y+2x) - 8xg''(y+2x) \\ &\quad + 4f''(y+2x) + 4xg''(y+2x) = 0. \quad ✓ \end{aligned}$$

Problem 5: Reduce the elliptic equation

(**)
$$u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$$ to the form

$v_{xx} + v_{yy} + cv = 0$ by a change of dependent variable

$u = ve^{ax+by}$ and then a change of scale $y' = \gamma y$.

Solution: Letting $u = ve^{\alpha x + \beta y}$ we have:

$$u_x = e^{\alpha x + \beta y}(\alpha v + v_x), \quad u_y = e^{\alpha x + \beta y}(\beta v + v_y),$$

$$u_{xx} = e^{\alpha x + \beta y}(\alpha^2 v + 2\alpha v_x + v_{xx}), \quad u_{yy} = e^{\alpha x + \beta y}(\beta^2 v + 2\beta v_y + v_{yy}).$$

Plugging into the PDE:

$$e^{\alpha x + \beta y}\left[v_{xx} + 3v_{yy} + (2\alpha - 2)v_x + (6\beta + 24)v_y + (\alpha^2 + 3\beta^2 - 2\alpha + 24\beta + 5)v\right] = 0.$$

taking $\alpha = 1$ and $\beta = -4$
makes these coefficients vanish
and we get

$1 + 3(16) - 2 - 96 + 5$

$1 + 48 - 2 - 96 + 5$

$$v_{xx} + 3v_{yy} - 44v = 0.$$

want a change of variables
that will make this coefficient 1.

Take $y' = \frac{1}{\sqrt{3}}y \Rightarrow v_{y'y'} = 3v_{yy} \Rightarrow$

$v_{xx} + v_{y'y'} - 44v = 0$. Hence, we reduced the elliptic PDE (**) into the desired form.