P. 52 Problem 2 Solve the diffusion equation with the initial condition $\phi(x) = 1$ for $x > 0$ and $\phi(x) = 3$ for $x < 0$.

Solution Using eq. (8) on p.49, we have:

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \left[\int_{-\infty}^{0} 3e^{-(x-y)^2/4kt}\, dy + \int_{0}^{\infty} e^{-(x-y)^2/4kt}\, dy \right]$$

Since our goal is to express u in terms of the error function on p. 51, it is natural to make the following change of variables: $u = \frac{(x-y)}{2\sqrt{kt}}$, $du = -\frac{dy}{2\sqrt{kt}}$. Then, the solution becomes:

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \cdot -2\sqrt{kt} \left[\int_{+\infty}^{x/2\sqrt{kt}} 3e^{-u^2}\, du + \int_{x/2\sqrt{kt}}^{-\infty} e^{-u^2}\, du \right]$$

$$= \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^{x/2\sqrt{kt}} e^{-u^2}\, du + \int_{x/2\sqrt{kt}}^{+\infty} 3e^{-u^2}\, du \right]$$

$$= \frac{1}{\sqrt{\pi}}\left[\int_{-\infty}^{0} e^{-u^2}\,du + \int_{0}^{x/2\sqrt{\kappa t}} e^{-u^2}\,du + \int_{0}^{+\infty} 3e^{-u^2}\,du - \int_{0}^{x/2\sqrt{\kappa t}} 3e^{-u^2}\,du\right]$$

$$= \frac{1}{\sqrt{\pi}}\left[\frac{1}{2}\int_{-\infty}^{+\infty} e^{-u^2}\,du + \frac{3}{2}\int_{-\infty}^{+\infty} e^{-u^2}\,du + \frac{\sqrt{\pi}}{2}\,\mathcal{E}rf\left(\frac{x}{2\sqrt{\kappa t}}\right) - \frac{3\sqrt{\pi}}{2}\,\mathcal{E}rf\left(\frac{x}{2\sqrt{\kappa t}}\right)\right]$$

here we use that e^{-u^2} is even and so $\frac{1}{2}\int_{-\infty}^{+\infty} e^{-u^2}\,du = \int_{0}^{+\infty} e^{-u^2}\,du = \int_{-\infty}^{0} e^{-u^2}\,du.$

definition of error function

$$= \frac{1}{\sqrt{\pi}}\left[2\int_{-\infty}^{+\infty} e^{-u^2}\,du - \sqrt{\pi}\,\mathcal{E}rf\left(\frac{x}{2\sqrt{\kappa t}}\right)\right]$$

here we use $\int_{-\infty}^{+\infty} e^{-u^2}\,du = \sqrt{\pi}.$

$$= \frac{1}{\sqrt{\pi}}\left[2\cdot\sqrt{\pi} - \sqrt{\pi}\,\mathcal{E}rf\left(\frac{x}{2\sqrt{\kappa t}}\right)\right]$$

$$= \boxed{2 - \mathcal{E}rf\left(\frac{x}{2\sqrt{\kappa t}}\right)}$$

P.52 Problem 4 Solve the diffusion equation if $\phi(x) = e^{-x}$ for $x > 0$ and $\phi(x) = 0$ for $x < 0$.

Solution Again, using eq.8 on p.49 we have:

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^{+\infty} e^{-(x-y)^2/4kt} e^{-y} \, dy$$, we continue

exactly as on p.52, the exponent is:

$$\frac{-x^2 - 2xy + y^2 + 4kty}{4kt}$$. Completing the square in the

y variable, it is $$-\frac{(y + 2kt - x)^2}{4kt} + kt - x.$$

Let $u = (y + 2kt - x)/\sqrt{4kt}$, $du = dy/\sqrt{4kt}$. Then,

$$u(x,t) = \frac{e^{kt-x}}{\sqrt{\pi}} \int_{\frac{(2kt-x)}{\sqrt{4kt}}}^{+\infty} e^{-u^2} du = \frac{e^{kt-x}}{\sqrt{\pi}} \int_0^{+\infty} e^{-u^2} du - \frac{e^{kt-x}}{\sqrt{\pi}} \int_0^{(2kt-x)/\sqrt{4kt}} e^{-u^2} du$$

$$= \frac{e^{kt-x}}{\sqrt{\pi}} \cdot \frac{1}{2} \int_{-\infty}^{+\infty} e^{-u^2} du - \frac{e^{kt-x}}{2} \mathcal{E}rf\left(\frac{2kt-x}{\sqrt{4kt}}\right)$$

↳

$$= \frac{e^{kt-x}}{\cancel{\sqrt{\pi}}} \cdot \frac{1}{2} \cdot \cancel{\sqrt{\pi}} - \frac{e^{kt-x}}{2} \operatorname{Erf}\left(\frac{2kt-x}{\sqrt{4kt}}\right)$$

$$= \boxed{\frac{e^{kt-x}}{2}\left[1 - \operatorname{Erf}\left(\frac{2kt-x}{\sqrt{4kt}}\right)\right].}$$

P. 52 Problem 8 Show that for any fixed $\delta > 0$,

$$\max_{\delta \le |x| < \infty} S(x,t) \to 0 \quad \text{as} \quad t \to 0.$$

Solution $\displaystyle \max_{\delta \le |x| < \infty} S(x,t) = \max_{\delta \le |x| < \infty} \frac{1}{2\sqrt{\pi k t}} e^{-x^2/4kt}$

$$\le \frac{1}{2\sqrt{\pi k t}} e^{-\delta^2/4kt}$$

since $t > 0$, $S(x,t)$ is decreasing in x.

So, $\displaystyle \lim_{t \to 0} \left(\max_{\delta \le |x| < \infty} S(x,t) \right) \le \lim_{t \to 0} \frac{1}{2\sqrt{\pi k t}} e^{-\delta^2/4kt}$

make substitution $\tilde{t} = \frac{1}{4kt}$, then as $t \to 0$, $\tilde{t} \to \infty$

$$= \lim_{\tilde{t} \to \infty} \frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{\tilde{t}}}{e^{\delta^2 \tilde{t}}}$$

∞/∞ so L'H rule gives:

$$= \lim_{\tilde{t} \to \infty} \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{\sqrt{\tilde{t}}\,\delta^2 e^{\delta^2 \tilde{t}}} = 0. \quad \checkmark$$

P.54 Problem 18 Solve the heat equation with convection:

$u_t - ku_{xx} + Vu_x = 0$ For $-\infty < x < \infty$ with $u(x,0) = \phi(x)$,

where V is a constant. (Hint: Go to a moving frame of reference by substituting $y = x - Vt$).

Solution Following the hint, consider

$w(y,t) = u(y+Vt, t)$. Then, using the chain rule:

$w_t = u_x \cdot V + u_t$ and $w_y = u_x$

$w_{yy} = u_{xx}$.

So, $w_t - w_{yy} = u_t - u_{xx} + Vu_x = 0$ assuming that

u satisfies the heat equation with convection. Moreover,

$w(y,0) = u(y,0) = \phi(y)$. So,

$w(y,t) = \int_{-\infty}^{\infty} S(y-s, t)\phi(s)\,ds$. So,

where S is the diffusion kernel from p.50.

$$\boxed{u(x,t) = w(x-Vt, t) = \int_{-\infty}^{\infty} S(x-Vt-s, t)\,\phi(s)\,ds}$$