Bell ringing and group theory

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When the bell is nearly upside down, a little tug on the rope adds enough momentum to ensure that, in a few seconds, it swings all the way in the other direction so that its near upside down again. From then on, the bell can be kept ringing with comparatively small tugs on the rope at the appropriate times.

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The fact that only small adjustments can be made in the timing of when each bell rings means that the position of a bell in the extent can only be changed by one place at a time.

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- (Optional:) want to ring all possible permutations of n bells, starting with the trivial permutation.
- Don't want any repetitions.
- From one change to the next, any bell can move by at most one position in its order of ringing.

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Such a sequence of permutations is called an **extent**.

Example: Plain Bob

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- By convention, bell ringers are not permitted any memory aids such as sheet music.
- This means that a serious bell ringer must effectively recite a sequence of several thousand numbers, one every two seconds, and to translate this sequence into perfect bell ringing.
- If we have a large number of bells, it is not obvious how to construct an extent that obeys the bell ringing rules.

One way to construct a sequence

3 bells
123
213
231
321
312
132
(123)

123	231	312
123	231	312
123	231	312
123	231	312
213	321	132
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4 bells

1234	2314	3124
1243	2341	3142
1423	2431	3412
4123	4231	4312
4213	4321	4132
2413	3421	1432
2143	3241	1342
2134	3214	1324
		(1234)

More fun facts

bells	name	changes in extent	time required
3	Singles	6	12 seconds
4	Minimus	24	48 seconds
5	Doubles	120	4 minutes
6	Minor	720	24 minutes
7	Triples	5 040	2 hours 48 minutes
8	Major	40 320	22 hours 24 minutes
9	Caters	362 880	8 days 10 hours
10	Royal	3 628 800	84 days
11	Cinques	39 916 800	2 years 194 days
12	Maximus	479 001 600	30 years 138 days
16		20 922 789 888 000	1 326 914 years

An extent on eight bells was rang only once, at the Loughborough Bell Foundry in 1963. The ringing began at 6.52am on July 27, and finished at 12.50am on July 28, after 40,320 changes and 17 hours 58 minutes of continuous ringing

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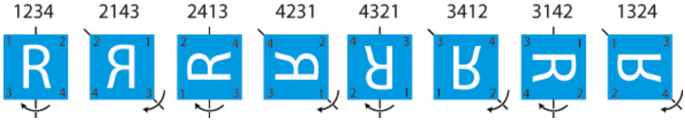
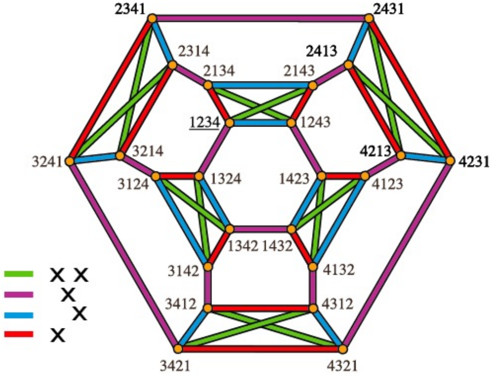
- look at the separation of the Plain Bob extent into 3 columns. They are cosets of D_4 as a subgroup of S_4.

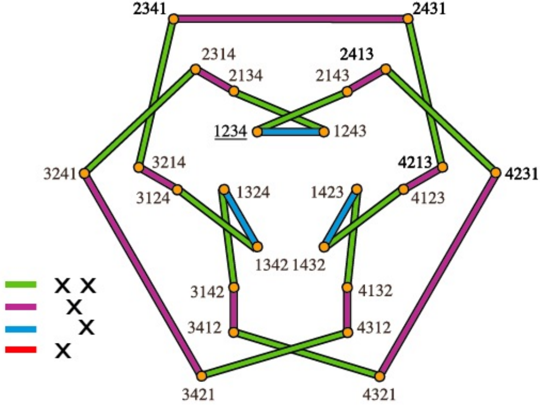


Graph out of bell ringing

- Think of permutations as **vertices of a graph**.
- Two vertices are connected by an edge if there is a permitted transition (according to bell ringers) that transforms one change into the other. Here what it looks like for Plain Bob:



- An extent is simply a path in this graph, visiting each of the vertices exactly once, and returning to the beginning vertex. Such tours are called **Hamiltonian cycles**.
- For Plain Bob, this path looks like that:



Cayley Graph

Let G be a group, and let S be a generating set of elements.

Definition

Let $Cay(G,S)$ be the colored directed graph having G as the set of vertices, and for any $s \in S$ there is an edge going from g to gs, and any such edge is colored into a unique color c_s corresponding to $s \in S$.

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- D_4 with generators r_{90} (rotation by 90°) and s_h (vertical reflection)?

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- **Theorem.** The graph $Cay(\{(1,0),(0,1)\} : \mathbb{Z}/m \times \mathbb{Z}/n)$ has a Hamiltonian circuit when n divides m.